Studies on

Quantum Field Theory and Statistical Mechanics

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Abstract of the Dissertation

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This dissertation is a summary of research in various areas of theoretical physics and is divided into three parts. In the first part, quantum fluctuations of the recently proposed superconducting cosmic strings are studied. It is found that vortices on the string world sheet represent an important class of fluctuation modes which tend to disorder the system. Both heuristic arguments and detailed renormalization group analysis reveal that these vortices do not appear in bound pairs but rather form a gas of free vortices. Based on this observation we argue that this fluctuation mode violates the topological conservation law on which superconductivity is based .

Anomalies and topological aspects of supersymmetric quantum field theories are studied in the second part of this dissertation. Using the superspace formulation of the $N = 1$ spinning string, we obtain a path integral measure which is free from the world–sheet general coordinate as well as the supersymmetry anomalies and therefore determine the conformal anomaly and critical dimension of the spinning string . We also apply Fujikawa's formalism to compute the chiral anomaly in conformal as well as ordinary supergravity. Finally, we give a Noether–method construction of the supersymmetrized Chern–Simons term in five dimensional supergravity.

In the last part of this dissertation, the soliton excitations in the quarter–filled Peierls–Hubbard model are investigated in both the large and the small U limit. For a strictly one dimensional system at zero temperature, we find that solitons in both limits are in one–to–one correspondence, while in the presence of weak three dimensional couplings or at finite temperature , the large U systems differ qualitativly from the small U systems in that the spin associated with the solitons ceases to be a sharp quantum observable.

to my father

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PART A

Introduction

Recent developments in cosmology are clearly marked by the close interaction with both particle physics and statistical mechanics. According to the standard Grand Unification Theories, the interaction between elementary particles is described by a gauge symmetry group which is an exact symmetry only at sufficiently high temperature or in the very early stage of the universe. As the universe expands and cools down, it undergoes a series of phase transitions, and this symmetry is spontaneously broken below the transition temperature [1]. These phase transitions in the early universe give rise to topological defects such as domain walls, strings and monopoles [2]. While the abundence of domain walls and monopoles has disastrous cosmological consequences [3], cosmic strings can lead to interesting cosmological effects. A cosmic string originating in symmetry breaking at a scale of $10^{16} GeV$ has a mass density of $10^{22} g/cm$. Based on this huge mass density, it has been speculated that such strings can generate density fluctuations large enough to explain galaxy formation [4] and produce a number of observational effects by acting like a gravitational lense [5]. Cosmic strings and their astrophysical consequences are reviewed extensively by Vilenkin in Ref. [6].

Recently, another class of cosmic strings has been proposed by Witten [7]. He observed that under certain conditions, a Higgs field coupled to the electromanetic field could develop a nonvanishing expectation value in the core of the cosmic string and argued that such strings should be superconducting. In contrast to the usual type of cosmic strings which only have gravitational effects, superconducting strings can produce spectacular effects based on their electromagnetic properties. In particular, they might be observable as a cosmic synchrotron radiation source. More recently, Osteriker, Thompson and Witten [8] explored the effects of superconducting strings on galaxy formation. They argued that the electromagnetic radiation of an oscillating current–carrying string loop may substantially exceed its gravitational radiation and such string loops could heat their surroundings, generating large, dense spherical shells of gas and therefore resulting the formation of galaxies.

A cosmic string is only superconducting if there is a conserved topological charge [7]

$$N = \frac{1}{2\pi} \oint \frac{d\theta}{dl} dl$$

where θ is the phase of the charged Higgs field in the core of the string. Our paper is a critical investigation of the fact whether N is truely conserved. Relevant to this question is a statistical property of the vortices on the string world sheet: Whether vortices are free or bound into pairs. Similar investigations have been carried out by Kosterlitz and Thouless [9] on superfluid films where they found that both the creation energy and the entropy of a single vortex diverge logarithmically with the size of the system. From this observation they argued that there exists a phase transition at which bound vortices dissociate into free ones and that above the transition temperature superfuidity is destroyed. In our case, we find that the vortex creation energy diverge less rapidly as the entropy and this leads us to conclude that vortices always appear in form of a free plasma. This simple but heurestic argument is indeed supported by a detailed renormalization group calculation which is included in the appendix of this chapter. Based on these observations we argue that the cosmic strings of this kind are actually *not* superconducting.

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Quantum Fluctuations of the Superconducting Cosmic String

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Abstract

Quantum fluctuations of the proposed superconducting string with Bose charge carriers are studied in terms of the vortices on the string world sheet. We find that they appear in the form of free vortices rather than as bound pairs. This fluctuation mode violates the topological conservation law on which superconductivity is based.

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A charged Higgs boson coupled to the electromagnetical field can obviously not have a nonvanishing vacuum expectation value, since then the vacuum would be a superconducting medium and the photon would be massive. However, as pointed out recently by Witten [1], the expectation value of this Higgs field could have an inhomogeneous distribution, which vanishes everywhere except on a lower dimensional object such as a string. He showed that under certain conditions, the cosmic string created in the early universe could indeed support such a distribution of the Higgs field and argued that such string could then be superconducting. Cosmological consequences based on this observation were explored in Ref. [2].

This paper discusses the quantum fluctuations of such a superconducting string. As common in many systems with broken symmetries, there are topological excitations which tend to disorder the system. These fluctuations are particularly important for two dimensional systems with a $O(2)$ symmetry. For example, the topological vortices in a superfluid film can be created via thermal fluctuations. At low enough temperatures, these vortices can only occur in tightly bound pairs and do not change the long distance characteristics of the system. However, as the temperature is increased to a critical value, these pairs dissociate into free vortices and they drive a phase transition which destroys the superfluidity. The phase transition driven by the vortices is commonly known as the Kosterlitz- Thouless transition [3]. Similar phase transitions also occur in superconducting films under more restrictive conditions [4]. We investigate the effect of quantum fluctuations in terms of the vortices on the world sheet of the superconducting string in a similar fashion. Unlike the instantons of the abelian Higgs model in $1+1$ dimensions [5], these vortices are not localized in space-time, the action (or let us call it the creation energy) of a single vortex diverges with the size of the system and vortices interact with each other through a long ranged potential. However, due to the effect of the electromagnetical screening, we find that the creation energy of a single vortex diverges less rapidly than the entropy, (in fact, the creation energy $\sim lnlnL$, while the entropy $\sim lnL$, L being the linear size of the system). Free energy therefore favors the creation of free vortices. In addition to this observation, we have performed a renormalization group analysis (similar to that of Ref. [6]) which incorporates the long ranged interaction of the vortices and have found that indeed a condensate of free vortices with finite density is present in the ground state. The density of the free vortices serves as a "disorder parameter"

[7] of the system and its nonvanishing means that the system is in a disordered phase. From this we therefore argue that the proposed superconducting string does not sustain persistent currents like a bulk superconductor. In a bulk superconductor, the "disordering agent" is the Abrikosov string [8]. In the superconducting phase, the density of these strings vanishes in the thermodynamical limit and the supercurrent is stable.

Let us start by recalling some basic facts about the proposed superconducting string with Bose charge carriers [1]. One considers a $U(1) \times \tilde{U}(1)$ gauge field theory, with gauge fields A_μ and R_μ and Higgs fields σ and ϕ, interacting according to the following Lagrangian:

$$L = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}R_{\mu\nu}^2 + |D_\mu\sigma|^2 + |D_\mu\phi|^2 - \frac{1}{8}\lambda(|\phi|^2 - \mu^2)^2 - \frac{1}{4}\tilde{\lambda}|\sigma|^4 - f|\sigma|^2|\phi|^2 + m^2|\sigma|^2 \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $R_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu$, $D_\mu\sigma = (\partial_\mu + ieA_\mu)\sigma$ and $D_\mu\phi = (\partial_\mu + igR_\mu)\phi$.

In the range of parameters given by $\mu^2, m^2 > 0$ and $f\mu^2 - m^2 > 0$, the $\tilde{U}(1)$ symmetry is broken ($| < \phi > | = \mu$), while the electromagnetic $U(1)$ symmetry is unbroken ($| < \sigma > | = 0$). However, there exists a classical solution to the broken $\tilde{U}(1)$ theory in forms of a string [8], where the Higgs field ϕ vanishes in the core of the string and approaches μ at infinity. In this case, as shown in Ref. [1], the Higgs field σ has the opposite behavior: it vanishes everywhere except in the core of the string. Provided that the amplitude of the σ field in the string varies smoothly over the range of the coherence length $1/m$, (m being the mass parameter of the σ field, see(1)) , the dynamics of the low lying phase excitations $\theta(x_3, x_0)$ of the σ field can be treated by considering the following effective action:

$$I = \int d^4x[-\frac{1}{4}F_{\mu\nu}^2 + K\delta^2(x_1, x_2)\,(\partial_i\theta + eA_i)^2] \quad (2)$$

where $\mu = 0, 1, 2, 3$ and $i = 0, 3$ and without loss of generality, we have assumed that the string is directed along the third axis. K is a constant estimated to be $K \approx 1/\tilde{\lambda}$.

The way to see whether the string is superconducting is to observe that the model defined by (2) admits a topological invariant

$$N = \frac{1}{2\pi}\oint dx_3 \frac{\partial\theta}{\partial x_3} \quad (3)$$

Since θ is an angular parameter, N so defined is always an integer. In Ref. [1] it was shown that in a sector with nonzero N, the ground state is current carrying; if N is conserved, the current can not decay and the string is therefore superconducting. However, even though the topologically trivial phase fluctuations can not change the quantum number N, there are topological vortex excitations which violate the conservation of N, as was originally remarked in Ref. [1]. A vortex on the string world sheet is a singularity where the amplitude of the σ field vanishes and the phase becomes ill-defined. A vortex with unit vorticity can change N by one unit. Therefore, N is only conserved if these vortices form tightly bound objects with zero net vorticity, since at large distances these objects are indistinguishable from the topologically trivial phase fluctuations. The question about the conservation of N is hence reduced to a statistical mechanics problem of the vortex gas: Do the vortices appear in the form of tightly bound objects or do they form a gas of free vortices?

As we shall study the imaginary time propagation of the string, let us consider the Euclidean version of (2) by a Wick rotation ($x_0 = ix_4$). In the Lorentz gauge $\partial^\mu A_\mu = 0$, the components A_1 and A_2 are decoupled and can be set to zero without loss of generality. The Euclidean version of (2) is then given by

$$I_E = \int d^4x[-\frac{1}{2}A_i \Delta A_i + K\delta^2(x_1, x_2)(\partial_i\theta + eA_i)^2] \tag{4}$$

where $\Delta \equiv \sum_{\mu=1}^4 \frac{\partial^2}{\partial x_\mu^2}$ and from now on, the index i refers to $i = 3, 4$.

In order to study the effective interaction of the vortices, we first integrate out A_i to get an effective action for θ. The equation of motion of A_i is given by

$$\Delta A_i = 2Ke\delta^2(x_1, x_2)(\partial_i\theta + eA_i) \equiv J_i \tag{5}$$

or in Fourier space

$$A_i(q) = -\frac{1}{q^2}J_i(q) \tag{6}$$

Let us denote $q \equiv (q_1, q_2, q_3, q_4)$ and $p \equiv (q_3, q_4)$. Due to the delta function in the definition of $J_i(x)$, $J_i(q)$ is actually a function $J_i(p)$ of q_3 and q_4 only. Integrating (6) over q_1 and q_2 yields

$$\int dq_1 dq_2 A_i(q) = -J_i(p) \int dq_1 dq_2 \frac{1}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \approx 2\pi J_i(p) ln(|p|r_0) \tag{7}$$

where r_0 is the thickness of the string which serves as an ultraviolet cutoff of the integral. Defining $A_i(p) \equiv \int d^4x e^{-ixq} A_i(x)\delta^2(x_1, x_2)$ we find that the left hand side of (7) is just

$(2\pi)^2 A_i(p)$. Therefore

$$2\pi A_i(p) = 2Ke[\Phi_i(p) + eA_i(p)]ln(|p|r_0) \tag{8}$$

where $\Phi_i \equiv \partial_i\theta$. Solving this equation we obtain

$$eA_i(p) = \frac{\alpha ln(|p|r_0)}{1-\alpha ln(|p|r_0)}\Phi_i(p) \quad and \quad J_i(p) = \frac{2Ke}{1-\alpha ln(|p|r_0)}\Phi_i(p) \tag{9}$$

where $\alpha \equiv Ke^2/\pi$

These relations are to be contrasted with the usual type of Higgs mechanism, where $A_i(p)$ has a simple pole at the mass of the vector boson or the inverse London penetration depth. This usual behavior is nothing but the Meissner screening effect. In our case, however, the "photon mass" cannot result simply because both coupling constants K and e are dimensionless. The physical reason behind this is rather clear: The electromagnetical field extends over four dimensional space-time while the matter field is restricted to the two dimensional world sheet. The effect of the induced currents in screening the electromagnetical field is therefore much weaker, and in fact only logarithmic. The singularity in (9) appears at momenta greater than the physical cutoff $1/r_0$ and so will not concern us here. From (9) we can easily find the effective action for θ:

$$I_E = K\int \frac{d^2p}{(2\pi)^2}\frac{p^2}{1-\alpha ln(|p|r_0)}\theta(p)\theta(-p) \tag{10}$$

A vortex (with infinitesimal core radius) located at $\bar{x}$ is defined by

$$\nabla^2\theta(x) = -2\pi n\delta^2(x-\bar{x}) \tag{11}$$

or by the fact that θ changes by $2\pi n$ along any contour enclosing $\bar{x}$:

$$\oint d\vec{x}\vec{\nabla}\theta = 2\pi n \tag{12}$$

The integer n is called the vorticity. The loop integral in (12) can be taken along the boundary of the string world sheet . Identifying the both ends of the string and using (3) and (12) we find that

$$N(x_4 = \infty) - N(x_4 = -\infty) = n \tag{13}$$

therefore, a vortex with vorticity n changes the winding number N by n units.

Given the effective action (10), the action of a single vortex is very easy to compute. From (11) we have $\theta(p) = 2\pi n e^{-ip\bar{x}}/p^2$. Substituting this into (10) we obtain

$$I_0 = 2\pi n^2 K \int \frac{dp}{p} \frac{1}{1 - \alpha ln(pr_0)} \approx \frac{2\pi^2 n^2}{e^2} ln(1 + \alpha ln \frac{L}{r_0}) \tag{14}$$

The ultraviolet cutoff of this integral is the core radius of the vortex $a \sim 1/m$, the scale over which the amplitude of the σ field varies, while the infrared cutoff is simply L ,the linear extention of the string world sheet. We have assumed that a and r_0 are of comparable length. The way I_0 diverges with L is a particular consequence of the effective logarithmic screening. If perfect Meissner screening would take place (like in the abelian Higgs model in $1+1$ dimensions [5]), I_0 would remain finite. If there were no screening at all, I_0 would diverge like lnL [3]. In our case, while the creation energy I_0 of a free vortex diverges as $lnlnL$, its "entropy" is simply $S = ln(\frac{L}{a})^2$. The entropy term arises in the path integral in the integration over the zero mode, namly the position of the vortex. The contribution of a single vortex to the partition function is

$$Z = D_0 D e^{-I_0} \tag{15}$$

where D_0 is the zero mode contribution, $D_0 = (\frac{L}{a})^2$ and D is the fluctuation determinant. Exponetiating D_0 we obtain the entropy term.

Comparing I_0 with S we see that in the large L limit, the entropy term always overweights the creation energy. The free energy $F = -lnZ = I_0 - S$ is therefore always lowered by creating vortices. This simple argument strongly suggests that a gas of free vortices is the favorable configuration. However, this argument is only heuristic in the sense that it only involves a single vortex, while vortices actually have long ranged interactions. In Ref. [6] Kosterlitz performed a detailed renormalization group analysis to support a similar simple argument for the two dimensional XY model. In order to carry out a similar analysis, let us first find the partition function of the vortex gas. A multivortices configuration is defined by

$$\nabla^2 \theta(x) = -2\pi \sum_i n_i \delta^2(x - x_i) \tag{16}$$

where n_i is the vorticity and x_i is the location of the $i-th$ vortex. From (16) it follows that $\theta(p) = \sum_i 2\pi n_i e^{-ipx_i}/p^2$. Substituting this into (10) we obtain the partition function of a vortex gas:

$$I = (2\pi)^2 K \sum_{i,j} n_i n_j G(x_i - x_j) \tag{17}$$

where

$$G(x) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{1 - \alpha ln(|p|r_0)} \frac{e^{-ipx}}{p^2} \tag{18}$$

Only overall neutral configuration $\sum_i n_i = 0$ contribute to the partition function, since nonneutral configurations have infinite action associated with them. For $\alpha = 0$, this reduces to the partition function of a two dimensional Columb gas, interacting with a logarithmic potential.

In order to find out about the presence of free vortices, one adds a chemical potential term $-lny \sum_i n_i^2$ to (17) which controls the density of vortices. Our question can therefore be formulated in a more precise way : Is the added chemical potential term a relevant or an irrelevant perturbation in the sense of the renormalization group flow ? The partition function (17) only reveals the degrees of freedom at the scale of the ultraviolet cutoff a; at this scale, it is impossible to distinguish tightly bound vortices from the free ones. To see their differences, one has to go to larger and larger scales by the procedure of coarse graining, or integrating out small distance fluctuations. If vortices are tightly bound, coarse graining would decrease their effective density , since at distances larger than their separation, they are indistinguishable from topologically trivial phase fluctuations; y and therefore the effective density of vortices scales to zero. On the other hand, if there is a gas of *free* vortices present, coarse graining will only increase their effective density. Vortices becomes the dominant configurations at large distances.

We have carried out a renormalization group analysis to the lowest order in perturbation theory in y, with only vortices of unit vorticity included. The calculation follows the approach of Ref. [9]. The idea is to map the partition function of the vortex gas to a field theoretical model by means of a duality transformation and then carry out a standard momentum shell integration. (For details of the calculation, see Ref. [10]). To the *lowest* order, the renormalization group equation we obtained is of the following form:

$$dK = -\frac{K^2 e^2}{\pi}\frac{da}{a} \quad and \quad dy = -y\frac{da}{a}\left(\frac{2\pi K}{1+\alpha ln\frac{a}{r_0}} - 2\right) \tag{19}$$

where a is the renormalization scale. Constants e and r_0 are unrenormalized. This renormalization group equation posses a line of fixed points $y = 0$. This agrees with the common knowledge that for $y = 0$, only topologically trivial phase fluctuations are present and they lead to algebraically decaying correlation functions in two dimensions. For $e = 0$, (recall that $\alpha = \frac{Ke^2}{\pi}$), (19) reduces to the Kosterlitz-Thouless case:

$$dy = -y\frac{da}{a}(2\pi K - 2), \qquad y \propto a^{2-2\pi K} \tag{20}$$

This indicates that for $K < 1/\pi$, the line of fixed points is infrared unstable, and the perturbation is relevant, while for $K > 1/\pi$ the line of fixed points is stable. In our case, from (19)

$$\frac{1}{K} = \frac{1}{K_0} + \frac{e^2}{\pi}ln\frac{a}{r_0} \quad and \quad y \propto a^2\left(1 + \frac{2K_0 e^2}{\pi}ln\frac{a}{r_0}\right)^{-\pi^2/e^2} \tag{21}$$

where K_0 is the value of K at the scale $a = r_0$. For large a, y always increases as one scales towards large distances and the entire line of fixed points is infrared unstable . Vortices are always relevant!

In conclusion we therefore find that the renormalization group analysis confirms the simple argument based on creation energy versus entropy : the string world sheet is indeed populated by a gas of free vortices and they represent the dominant fluctuations at large distances. This implies that N is not a conserved quantum number and different N sectors do communicate with each other. In the presence of dissipation, the system will relax until it reaches the absolute ground state with $N = 0$ which is not current carrying. In this sense we argue that the cosmic string with Bose charge carriers is not a superconductor. Similar analysis of the cosmic string with Fermi charge carriers is still under investigation.

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Appendix

In this appendix, we shall derive the renormalization group equation (19) paper **1** by following a similar approach of Kogut [10]. The main idea is to map the partition function of the vortex gas to a field theoretical model with well–defined propagators and vertices in order to perform the standard momentum shell integration. Let us denote the density of vortices by

$$m(x) = \sum_i n_i \, \delta^2(x - x_i) \qquad (A-1)$$

with x specifying the position on a two dimensional lattice with lattice spacing a. The partition function then becomes

$$Z = \sum_{m(x)} e^{-4\pi^2 K \sum_{x,x'} m(x)m(x')G(x-x') + lny \sum_x m^2(x)} \qquad (A-2)$$

with $G(x)$ defined in (18). We can perform a duality transformation on (A-2) by introducing a new Gaussian variable $\Phi(x)$, up to a multiplicativ constant, (A-2) can be represented by

$$Z = \int [D\Phi(x)] \sum_{m(x)} e^{\sum_x [\frac{1}{4\pi^2 K}\Phi(x)\Delta\Phi(x) + 2im(x)\Phi(x) + lny\ m^2(x)]} \qquad (A-3)$$

where

$$\Delta = \nabla^2 [1 - \alpha ln(|\nabla| r_0)] \qquad (A-4)$$

is a kinetic operator of the $\Phi(x)$ field which leads to a prpagator of the form $\frac{1}{p^2(1-\alpha ln(|p|r_0)}$ in the momentum space. For small y, only $m(x) = 0, \pm 1$ contribute significantly to the partition function, in which case the summation over $m(x)$ can be carried out in a trivial way:

$$\sum_{m(x)=0,\pm1} e^{2im(x)\Phi(x) + lny\ m^2(x)} = 1 + e^{lny + 2i\Phi(x)} + e^{lny - 2i\Phi(x)}$$

$$= 1 + 2ycos2\Phi(x) \approx e^{2ycos2\Phi(x)} \qquad (A-5)$$

Substituting this into (A-3) we obtain

$$Z = \int [D\Phi(x)] e^{\sum_x [\frac{1}{4\pi^2 K}\Phi(x)\Delta\Phi(x) + 2ycos2\Phi(x)]}$$

$$= \int [D\Phi(x)] e^{\int d^2x [\Phi(x)\Delta\Phi(x) + \mu cos 4\pi\sqrt{K}\Phi(x)]} \qquad (A-6)$$

where we have taken the continuum limit and rescaled $\Phi(x)$. μ is defined by $\mu = \frac{2y}{a^2}$. At $e = 0$, $\Delta = \nabla^2$ and (A-6) is nothing but the generating functional of the sine–Gordon model. At $e \neq 0$, extra nonlocal interactions are present. However, the quadratic part still gives a well–defined propagator and the renormalization group procedure can be carried out as usual.

To carry out the momentum shell integration, we cut off the $\Phi(x)$ field in (A-6) at a scale Λ:

$$\Phi_\Lambda(x) = \int^\Lambda \frac{d^2p}{(2\pi)^2} e^{ipx}\Phi(p) \qquad (A-7)$$

and we want to obtain a effectiv action for the field $\Phi_{\Lambda'}(x)$ cut off at a scale $\Lambda' < \Lambda$ by integrating out the fluctuations $h(x) = \Phi_\Lambda(x) - \Phi_{\Lambda'}(x)$ which is restricted in a momentum shell bounded by Λ' and Λ. We note that

$$Z_\Lambda = \int [D\Phi_{\Lambda'}(x)] e^{\int d^2x \Phi_{\Lambda'}(x)\Delta\Phi_{\Lambda'}(x)} Z' \qquad (A-8)$$

where

$$Z' = \int [Dh(x)] e^{\int d^2x [h(x)\Delta h(x) + \mu cos 4\pi\sqrt{K}(\Phi_{\Lambda'}(x)+h(x))]} \qquad (A-9)$$

To the lowest order in μ

$$Z' \approx 1 + \mu < \int d^2x cos 4\pi\sqrt{K}(\Phi_{\Lambda'}(x) + h(x)) > \qquad (A-10)$$

where the average of a operator $\mathbf{O}$ is defined by

$$< \mathbf{O} > \equiv \int [Dh(x)]\ \mathbf{O}\ e^{\int d^2x h(x)\Delta h(x)} \qquad (A-11)$$

Since only Gaussian integrations are involved, $(A-10)$ can be evaluated easily:

$$< cos 4\pi\sqrt{K}(\Phi_{\Lambda'}(x) + h(x)) > = A(0) cos 4\pi\sqrt{K}\Phi_{\Lambda'}(x) \qquad (A-12)$$

where

$$A(0) \equiv e^{-4\pi^2 K\tilde{G}(0)} \quad and \quad \tilde{G}(x) \equiv \int_{\Lambda'}^{\Lambda} \frac{d^2p}{(2\pi)^2} \frac{e^{-ipx}}{p^2(1-\alpha ln(|p|r_0))} \qquad (A-13)$$

Therefore

$$Z' = e^{\mu A(0)\int d^2x cos 4\pi\sqrt{K}\Phi_{\Lambda'}(x)} \qquad (A-14)$$

and by substituting this result into $(A-8)$ we obtain

$$Z_{\Lambda} = \int [D\Phi_{\Lambda'}(x)] e^{\int d^2x [\Phi_{\Lambda'}(x)\Delta\Phi_{\Lambda'}(x) + \mu A(0) cos 4\pi\sqrt{K}\Phi_{\Lambda'}(x)]} \qquad (A-15)$$

After rescaling the cut–off Λ' back to the original cut-off Λ by a change of integration variables, we find that the renormalized partition function has the same functional form as the original one, but the new parameters are

$$\alpha' = \frac{\alpha}{1-\alpha ln(\Lambda'/\Lambda)} \quad , \quad \mu' = \mu A(0)(\Lambda'/\Lambda)^{-2} \quad and \quad e' = e \qquad (A-16)$$

The differential renormalization group equations in (19) of paper **1** easily follow from $(A-16)$ by letting $\Lambda' = \Lambda - d\Lambda$ and converting the momentum space cut–off Λ into the real space cut–off a.

PART B

Introduction

Anomalies and topology have played a unique role in the recent developments of particle physics. Since the discovery of chiral anomalies about two decades ago in the investigation of π^0 decay and triangle graph [11], their impact on theoretical physics has reached far beyond the original scope. The subsequent developments have not only deepened our physical understanding and extended possible applications of anomalies [12], they also have revealed a profound connection between anomalies, differential geometry and topology [13]. In this part of the dissertation, we present investigations of chiral and conformal anomalies in supersymmetric quantum field theories by following the path integral formalism of Fujikawa [14], and construct the topological Chern–Simons term in five dimensional supergravity.

The connection between the path–integral and the chiral anomaly was first established by Fujikawa [14]. He noticed that a chiral transformation

$$\psi(x) \rightarrow e^{i\theta(x)\gamma_5}\psi(x)$$

in the path–integral

$$Z = \int [d\psi][d\overline{\psi}] e^{i\int d^4x \overline{\psi}\gamma^\mu(\partial_\mu + A_\mu)\psi}$$

gives rise to a Jacobian factor

$$J = e^{-2i\int d^4x \theta(x) "Tr" \gamma_5}$$

where the trace $"Tr"$ has to be defined by a careful regularization procedure. One way of regularizing the trace is to define a set of orthonormal eigenfunctions $\phi_n(x)$ satisfying

$$\gamma^\mu(\partial_\mu + A_\mu)\phi_n(x) = \lambda_n \phi_n(x)$$

$$\int d^4x \phi_n^\dagger(x)\phi_m(x) = \delta_{mn}$$

and damp the large eigenvalues by a Gaussian factor $e^{-\lambda_n^2/M^2}$:

$$"Tr"\gamma_5 \equiv \lim_{M\rightarrow\infty} \sum_n \phi_n^\dagger(x)\ \gamma_5\ e^{-\lambda_n^2/M^2}\phi_n(x)$$

Evaluating this sum one obtains a nonvanishing trace and therefore the anomalous divergence of the chiral current $j^5_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$:

$$\partial^\mu j^5_\mu = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

In paper **2**, we calculate the chiral anomaly in ordinary and conformal supergravity based on this approach. The chiral anomaly in ordinary supergravity has been computed before by other methods [15], while the chiral anomaly in conformal supergravity is a new result. These computations differ from the the original approach of Fujikawa in that both ordinary and conformal supergravity are gauge theories, the contributions from the various ghost fields must therefore be included properly. In conformal supergravity, additional problems arise because of its higher derivative nature. However, by using a theorem of Fujikawa [14] on the regulator independence of chiral anomalies, this problem can also be circumvented. Summarizing the results, we find that the chiral anomalies in ordinary and conformal supergravity are -21 and -20 times the anomaly of a real spin 1/2 field respectivly. The later result is confirmed in a calculation of the chiral anomalies in the $N = 4$ conformal supergravity by Römer and van Nieuwenhuizen [16].

Since anomalies are related to the Jacobian in the path integral, different choices of the path integral measure would lead to different anomalies. In gravitational theories, requiring the absence of anomalies in the general coordinate transformation uniquely specifies the functional measure [17]. Viewing the spinning string as a two dimensional field theory [18], it is natural to require the absence of the world sheet general coordinate as well as the supersymmetry anomalies. Both transfomations are part of the superspace general coordinate transformation in the superspace formulation [19]. In paper **3**, we indeed find a path integral measure which is free from the superspace general coordinate transformation anomalies. Having specified the path integral measure, we then proceed in calculating the conformal anomaly of the spinning string by applying Fujikawa's method to superspace and in agreement with previous results [18], [20], we also obtain $D = 10$ as the critical dimension of the spinning string.

The importance of the Chern–Simons terms in odd dimensional gauge theories was first pointed out by Schonfeld [21] and by Deser, Jackiw and Templeton [22]. They discovered the novel property that adding a Chern-Simons term

$$\epsilon^{\mu\nu\rho}\ Tr(F_{\mu\nu}A_\rho - \frac{2}{3}A_\mu A_\nu A_\rho)$$

to the usual Yang–Mills Lagrangian in three dimensions

$$Tr F^{\mu\nu}F_{\mu\nu}$$

leads to a gauge invariant mass which is quantized because of topological reasons. Chern–Simons terms also play an important role in supergravity and superstring theories. Chapline and Manton [23] noticed that the coupling of the Maxwell field to supergravity in ten dimensions as given by Bergshoeff, de Roo, de Wit and van Nieuwenhuizen [24] can easily be generalized to a coupling of the Yang–Mills field to supergravity by simply extending the combination

$$\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho$$

to the Chern–Simons term

$$\epsilon^{\mu\nu\rho}\ Tr(F_{\mu\nu}A_\rho - \frac{2}{3}A_\mu A_\nu A_\rho)$$

Based on this observation , Green and Schwarz [25] discovered the one–loop anomaly cancelation in superstring theories with gauge groups $SO(32)$ or $E_8 * E_8$ by adding to this Yang–Mills Chern–Simons term the corresponding gravitational Chern–Simons term

$$\epsilon^{\mu\nu\rho}\ Tr(R_{\mu\nu}\omega_\rho - \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho)$$

and including both terms into the action in a proper combination. Recently this gravitational Chern–Simons term has been supersymmetrized by Fré [26].

In paper **4**, we construct the supersymmetrization of the Chern–Simons term

$$\epsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}Tr(R_{\rho\sigma}\omega_\tau - \frac{2}{3}\omega_\rho\omega_\sigma\omega_\tau)$$

in five dimensional supergravity by using the time–honored Noether method. It is found that the supersymmetrization is only possible if one modifies the fermionic transformation law by adding purely bosonic terms. We speculate that if this also happens in supersymmetrizing the gravitational Chern–Simons term in ten dimensions, the compactification on Calabi–Yau background should be modified.

THE CHIRAL ANOMALY IN CONFORMAL AND ORDINARY SIMPLE SUPERGRAVITY IN FUJIKAWA'S APPROACH

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1. INTRODUCTION

In this contribution we shall reobtain the chiral anomaly of simple ordinary supergravity by means of Fujikawa's method[1)] as well as by the Pauli-Villars method. Then we shall present, as a new result, the axial anomaly for simple conformal supergravity.

Axial anomalies have been discussed extensively in recent articles. For supergravity, the issue is, as usual, more subtle than elsewhere, because one must fix gauges and add ghosts for the fermions in the loop. The axial anomal in simple ordinary supergravity has been calculated by various methods, see below. We begin by reobtaining the same result by means of the original Fujikawa method, since it is interesting in itself and will be used to illustrate certain aspects in the conformal computation. We show that using as regulator either the operator which is obtained directly from the classical action plus gauge fixing term, or simply the Dirac operator itself, yields the same result, which agrees with observations made in ref.[2]. We present the Pauli-Villars computation[3] because it most clearly shows which regulator should be used for a given anomaly. [As an example, we note that in a theory with only lef-handed spin 1/2 Dirac fermions the regulator is given by

$$\frac{1}{2}\,\not{D}(1+\gamma_5) + \frac{1}{2}\,\not{\partial}(1-\gamma_5) \tag{1}$$

since massive Pauli-Villars fields contains propagating left-and right-handed fields, while, however, only the (left-handed) fermion-loops must be regularized.]

The computations in conformal supergravity are based on the original papers of Kaku and Townsend and van Nieuwenhuizen[4]. We will also use some important results obtained by Fradkin and Tseytlin, who computed the β-function in N = 1,2,3,4 conformal supergravity[5]. It would be interesting to study the multiplet structure of the trace, chiral and other anomalies of conformal supergravity.

The gravitational spin 3/2 axial anomaly in four dimensions has been computed by various methods: by determining the eigenvalues of the relevant Hodge-de Rham operators[6], by a Feynman graph analysis (imposing gravitational conservation, the Adler-Rosenberg method[7], by zeta-function regularization (determining the a_2 coefficients by the coincidence limit method of Synge and DeWitt)[7], by the point splitting method[7], and by the topological method (determining the index of certain operators involved)[7]. Recently, Alvarez-Gaumé and Witten[2] computed the gravitational spin 3/2 axial anomaly in n dimensions, using a direct Feynman graph method (not by imposing gravitational conservation). They also gave a derivation of their results using a modificaiton of Fujikawa's method (by introducing a one-dimensional quantum-mechanical system whose Hamiltonian is equal to the exponent of Fujikawa's regulator, exp $(\not{D}/M)^2$).

In this article we intend to give yet another derivation of the gravitational spin 3/2 axial anomaly, namely following Fujikawa's original approach for the gravitational spin 1/2 axial anomaly[1]. For higher dimensions, this method becomes rather complicated, and the modification of ref.[2] is more suitable. However, the original Fujikawa method is rather simple in principle. As we shall show, no point splitting techniques need be used. We also point out that the method is quite similar to those Pauli-Villar's regularizations of the path-integral whose Jacobian for chiral transformations is unity; however, unlike in the Yang-Mills case, in the gravitational case one needs more than one regulator, due to the derivative couplings of gravity.

2. SPIN 1/2 CASE BY FUJIKAWA'S METHOD

Consider a complex massless spin 1/2 fermion ψ in a gravitational background described by vielbeins $e_\mu^{\ m}$. The generator for connected (i.e., one-particle irreducible, in this case) graphs reads

$$W = \int d\psi\, d\bar{\psi}\, \exp\, [-e\, \bar{\psi}\gamma^\mu D_\mu \psi] \tag{2}$$

Under a chiral transformation of integration variables

$$\psi \to (1 + \alpha\gamma_5)\psi, \quad \bar{\psi} \to \bar{\psi}(1 + \alpha\gamma_5) \tag{3}$$

the path-integral does not change. Hence, the Jacobian cancels the variation of the action.

$$-\ \text{"Tr"}(2\alpha\gamma_5) + \langle -e\, \bar{\psi}\gamma^\mu\gamma_5\psi\rangle \partial_\mu\alpha = 0 \tag{4}$$

To regularize the trace "Tr" over spacetime points and spinor indices, Fujikawa showed that any function $f(\not{D}^2)$ can be used, provided $f(0) = 1$. The most convenient choice is $f = \exp(\not{D}/M)^2$. Hence, using plane waves in a four-dimensional box

$$\begin{aligned} \text{"Tr"}(2\alpha\gamma_5) &= \Sigma \langle \vec{k}|\ e^{(\not{D}/M)^2}\ 2\alpha\gamma_5|\vec{k}\rangle \\ &\equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx}[\mathrm{tr}\ e^{(\not{D}/M)^2} 2\alpha\gamma_5]e^{ikx} \end{aligned} \tag{5}$$

where tr denotes the trace over spinor indices. By pulling the plane waves exp ikx to the left, the operator $\not{D}$ is replaced by $\not{D} + i\not{k}$, and one obtains.

$$"\mathrm{Tr}"(2\alpha\gamma_5) = \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr}(2\alpha\gamma_5) e^{-k^2/M^2} \exp(B/M^2)$$

$$B = 2ik \cdot D + D^2 + \frac{1}{4} R \tag{6}$$

In deriving this result, we used the cyclic identity for the Riemann tensor

$$\frac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu] = \frac{1}{8} \gamma^{\mu\nu} R_{\mu\nu}{}^{ab} \gamma_{ab} = \frac{1}{4} R \tag{7}$$

The evaluation of the integral is performed by expanding exp B/M^2, and only retaining terms which do not vanish when M^2 tends to infinity. The relevant terms are

$$"\mathrm{Tr}"(2\alpha\gamma_5) =$$

$$\int \frac{d^4k}{(2\pi)^4} e^{-k^2/M^2} \, \mathrm{tr} 2\alpha\gamma_5 \times \Big[1 + \frac{1}{1!} (D^2 + \frac{1}{4} R) M^{-2}$$
$$+ \frac{1}{2!} \{(2ik \cdot D)(2ik \cdot D) + (D^2 + \frac{1}{4} R)(D^2 + \frac{1}{4} R)\} M^{-4}$$
$$+ \frac{1}{3!} \{(2ik \cdot D)(2ik \cdot D)(D^2 + \frac{1}{4} R) + (2ik \cdot D)(D^2 + \frac{1}{4} R)(2ik \cdot D)$$
$$+ (D^2 + \frac{1}{4} R)(2ik \cdot D)(2ik \cdot D)\} M^{-6} + \frac{1}{4!} \{(2ik \cdot D)^4\} M^{-8}\Big] \tag{8}$$

Using $\int d^4k = M^4 \pi^2 \int d(k^2/M^2)(k^2/M^2)$ and $\int_0^\infty dy (\exp{-y}) y^n = \Gamma(n+1)$, we obtain

$$"\mathrm{Tr}"(2\alpha\gamma_5) =$$

$$\frac{2\alpha}{16\pi^2}\, \mathrm{tr}\gamma_5 \Big[M^2(D^2) + \frac{1}{2!}\ \{(-2)M^2D^2$$

$$+ (D^2 + \frac{1}{4}R)(D^2 + \frac{1}{4}R)\} + \frac{1}{3!}\ (-2)\{[D^2(D^2 + \frac{1}{4}R)$$

$$+ D_\mu(D^2 + \frac{1}{4}R)D^\mu + (D^2 + \frac{1}{4}R)D^2\} + \frac{1}{4!}\cdot\frac{6\times 16}{24}$$

$$\{D^2D^2 + D_\mu D^2 D^\mu + D_\mu D_\nu D^\mu D^\nu\}\Big] \tag{9}$$

The terms proportional to M^2 contain D^2, which contains four Dirac matrices since $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{mn}\gamma_{mn}$, but these terms are seen to cancel. The terms containing the scalar curvature R cancel, too, since they are given by

$$\frac{1}{2}\ (\frac{1}{4}\ D^2R + \frac{1}{4}\ RD^2) - \frac{1}{3}\ (\frac{1}{4}\ D^2R + \frac{1}{4}\ D_\mu RD^\mu + \frac{1}{4}\ RD^2) = \frac{1}{24}\ (D^2R) \tag{10}$$

which vanishes since $(D_\mu R) = (\partial_\mu R)$ so that the trace over $\gamma_5(D^2R)$ vanishes. In the remaining terms, those proportional to $(D^2)(D^2)$ cancel, too, and one is left with

$$\begin{aligned}\mathrm{Tr}"2\alpha\gamma_5" &= \frac{\alpha}{8\pi^2}\ \mathrm{tr}\gamma_5[-\frac{1}{3}\ D_\mu D^2 D^\mu + \frac{1}{6}\ \{D_\mu D^2 D^\mu + D_\mu D_\nu D^\mu D^\nu\}]\\ &= \frac{\alpha}{96\pi^2}\ \mathrm{tr}\gamma_5[D_\mu, D_\nu][D^\mu, D^\nu]\\ &= \frac{\alpha}{96\pi^2}\ \mathrm{tr}\gamma_5(\frac{1}{4}\ R_{\mu\nu}{}^{ab}\gamma_a\gamma_b)(\frac{1}{4}\ R^{\mu\nu cd}\gamma_c\gamma_d)\\ &= \frac{\alpha}{96\pi^2}\ (\frac{1}{4}\ \epsilon^{abcd}R_{\mu\nu ab}R^{\mu\nu}{}_{cd})\end{aligned} \tag{11}$$

This result as well as some of the intermediate steps are identical to[1], but we have avoided the use of point-splitting techniques in order to simplify the calculations. We will use the intermediate steps of this derivation when we discuss the spin 3/2 anomaly.

3. THE SPIN 3/2 CASE BY FUJIKAWA'S METHOD

Let us now consider the spin 3/2 case. The Rarita-Schwinger action for a real massless spin 3/2 field in an external gravitational field which satisfies the Einstein condition $R_{\mu\nu} = 0$ (necessary in order that the Rarita-Schwinger action be gauge-invariant by itself under $\delta\psi_\mu = D_\mu\epsilon$) reads, after adding the usual gauge fixing term

$$\begin{aligned}\mathcal{L} &= -\frac{e}{2}\bar{\psi}_\mu\gamma^{\mu\rho\sigma}D_\rho\psi_\sigma + \frac{e}{4}\bar{\psi}\cdot\gamma \not{D}\gamma\cdot\psi \\ &= \frac{e}{4}\bar{\psi}_\mu\gamma^\sigma\gamma^\rho\gamma^\mu D_\rho\psi_\sigma = \frac{e}{2}\bar{\psi}_\mu(\Gamma^\rho)^{\mu\sigma}D_\rho\psi_\sigma \end{aligned} \tag{12}$$

where $(\Gamma^\rho)^{\mu\sigma} \equiv \frac{1}{2}\gamma^\sigma\gamma^\rho\gamma^\mu$. Throughout we will suppress the spinor indices of the gravitino, but we will often explicitly exhibit the vector indices of the gravitino.

$$\text{"Tr"}(2\alpha\gamma_5) =$$

$$\int\frac{d^4k}{(2\pi)^4}\, e^{-ikx}\mathrm{tr}\,2\alpha\gamma_5\, \exp\left[\frac{(\Gamma^\mu D_\mu)(\Gamma^\nu D_\nu)}{M^2}\right]e^{ikx} \tag{13}$$

Using $\{\Gamma^\mu,\Gamma^\nu\}_{\rho\sigma} = 2g^{\mu\nu}g_{\rho\sigma}$ we get

$$\text{"Tr"}(2\alpha\gamma_5) =$$

$$\begin{aligned}&\int\frac{d^4k}{(2\pi)^4}\, e^{-ikx}\mathrm{tr}\,2\alpha\gamma_5 \exp M^{-2}\left(D^2 + \frac{1}{4}[\Gamma^\mu,\Gamma^\nu][D_\mu,D_\nu]\right)e^{ikx} \\ &= \int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\,2\alpha\gamma_5\, \exp\left(\frac{-k^2}{M^2}\right)\, \exp(B/M^2)\end{aligned} \tag{14}$$

where now

$$B = 2ik\cdot D + D^2 + C,\ C = \frac{1}{4}\,[\Gamma_\mu,\ \Gamma_\nu][D^\mu,\ D^\nu] \tag{15}$$

The evaluation of $[\Gamma^\mu,\Gamma^\nu][D_\mu,D_\nu]$ was given in[7)], but for completeness we will rederive it here.

$$C_{\lambda\rho} = \frac{1}{2}(\Gamma_\mu)_{\lambda\alpha}(\Gamma_\nu)_{\alpha\beta}[D_\mu,D_\nu]_{\beta\rho} = \frac{1}{8}(\gamma_\alpha\gamma_\mu\gamma_\lambda\gamma_\beta\gamma_\nu\gamma^\alpha)$$
$$\times\ [\frac{1}{4}\,R_{\mu\nu}{}^{ab}\gamma_a\gamma_b g_{\beta\rho} + R_{\beta\rho\mu\nu}I_s] \tag{16}$$

The symbol I_s denote the unit matrix in spinor space. Elementary Dirac matrix algebra leads to

$$\frac{1}{8}\,\gamma_\alpha\gamma_\mu\gamma_\lambda\gamma_\beta\gamma_\nu\gamma^\alpha = \frac{1}{2}(g_{\mu\lambda}g_{\beta\nu} - g_{\mu\beta}g_{\lambda\nu} + g_{\mu\nu}g_{\lambda\beta} - \epsilon_{\mu\lambda\beta\nu}\gamma_5) \tag{17}$$

since $\gamma_\alpha\gamma^{\rho\sigma}\gamma^\alpha = 0$ in four dimensions. Hence

$$C_{\lambda\rho} = (g_{\mu\lambda}g_{\beta\nu} - \frac{1}{2}\,\epsilon_{\mu\lambda\beta\nu}\gamma_5)(\frac{1}{4}\,R_{\mu\nu}{}^{ab}\gamma_{ab}g_{\beta\rho})$$
$$= \frac{1}{4}\,R_{\lambda\rho}{}^{ab}\gamma_{ab} + \frac{1}{4}\,R_{\lambda\rho}{}^{cd}\gamma_{cd} = \frac{1}{2}\,R_{\lambda\rho}{}^{ab}\gamma_{ab} \tag{18}$$

Instead of using $(\Gamma_\mu)_{\rho\sigma} = \frac{1}{2}\,\gamma_\sigma\gamma_\mu\gamma_\rho$ one could have used $(\Gamma_\mu)_{\rho\sigma} = g_{\rho\sigma}\gamma_\mu$ in the regulator. As claimed in ref.[2], "under certain broad assumptions" the result should be the same. In our case it is easy to see that this is true. Namely, not only is $\Gamma_{(\mu}\Gamma_{\nu)} = g_{\mu\nu}$ but also $C_{\lambda\rho}$ comes out the same. To see this, note that

$$C_{\lambda\rho} = \frac{1}{2}\,\gamma^{\mu\nu}g_{\lambda\beta}[D_\mu,D_\nu]_{\beta\rho} = \frac{1}{2}\,\gamma^{\mu\nu}[D_\mu,D_\nu]_{\lambda\rho}$$
$$= \frac{1}{2}\,\gamma^{\mu\nu}R_{\mu\nu\lambda\rho} \tag{19}$$

since the first term in $[D_\mu, D_\nu]$, namely $\gamma^{\mu\nu} R_{\mu\nu ab}\gamma^{ab}$, does not contribute, due to the cyclic identity, when $R_{\mu\nu} = 0$.

Thus the only difference in the expression for B for spin 3/2 as compared to spin 1/2 is that the term $\frac{1}{4}$ R is replaces by $\frac{1}{2} R_{\lambda\rho}{}^{ab}\gamma_{ab}$. In the spin 1/2 case, the $\frac{1}{4}$ R terms did not contribute at all. Looking at the expansion in (8), we see that the curvature term in the term with 1! still does not contribute, since it only has two Dirac matrices. The sum of all terms with only one curvature term is still as before, except that in this expression $\frac{1}{4}$ R is again to be replaced by $\frac{1}{2} R_{\lambda\rho}{}^{ab}\gamma_{ab}$. These terms again cancel, because they are equal to $((D^2)_{\lambda\alpha} \frac{1}{2} R_{\alpha\rho}{}^{ab})\gamma_{ab}$ which does not contain enough Dirac matrices to contribute (namely, it contains only the explicitly shown Dirac matrices γ_{ab}). Thus the only modification comes from the R^2 term in "Tr"$(2\gamma_5)$.

One finds for this contribution

$$R^2\text{-term} = \frac{2\alpha}{16\pi^2} \operatorname{tr} \gamma_5 \frac{1}{2!} \left(\frac{1}{2} R_{\mu\nu}{}^{ab}\gamma_{ab}\right)\left(\frac{1}{2} R^{\nu\mu cd}\gamma_{cd}\right)$$

$$= \frac{2\alpha}{16\pi^2} \left(-\frac{1}{2}\right)\left(\varepsilon^{abcd} R_{\mu\nu ab} R^{\mu\nu}{}_{cd}\right) \tag{20}$$

the terms without any curvature arrange themselves as in the spin 1/2 case, except that one must multiply the result by four, since one must trace over the vector indices as well as over the spinor indices of the gravitino. Thus the result from the gravitino plus gauge fixing terms to the spin 3/2 anomaly is

$$\text{"Tr"}\, 2\alpha\gamma_5 = \frac{\alpha}{96\pi^2} \left(\frac{1}{4} \varepsilon^{abcd} R_{\mu\nu ab} R^{\mu\nu}{}_{cd}\right)(4-24) \tag{21}$$

To obtain the complete anomaly, one must add the contribution from the Faddeev-Popov ghost (a complex spin 1/2 ghost with the same chiral weight as the gravitino) and of the Nielsen-Kallosh ghost (a real ghost, coming from the $\not{\partial}$ in the gauge fixing term, whose chiral weight is opposite to that of the gravitino). The sum of these contributions is (-1) and the total result is -21 times the anomaly for a real spin 1/2 field.

4. THE SPIN 1/2 CASE WITH PAULI-VILLARS REGULARIZATION

Let us now rederive the spin 1/2 axial anomaly, using Pauli-Villars regularization. We shall need two Pauli-Villars regulator fields, because, as we shall see, terms with the operator D^2 which vanish in the trace over spinor indices in the Yang-Mills case, do no longer vanish in the gravitational case, and are, moreover, divergent. These divergent terms cancel if one employs two regulator fields. We consider

$$Z = \int (d\psi d\bar{\psi})(d\chi_1 d\bar{\chi}_1)(d\chi_2 d\bar{\chi}_2) \exp S$$

$$S = \int d^4x \left[-e\bar{\psi}\gamma^\mu (D_\mu + iA_\mu\gamma_5)\psi + \sum_{i=1}^{2} \{ -e\bar{\chi}_i\gamma^\mu(D_\mu + iA_\mu\gamma_5)\chi_i - eM_i\bar{\chi}_i\chi_i \} \right] \tag{22}$$

We choose the chiral weights of χ_i such that <u>the measure is chirally invariant</u>

$$\delta\psi = \alpha\gamma_5\psi, \quad \delta\chi_i = -\frac{\alpha}{2}\gamma_5\chi_i \tag{23}$$

After a chiral transformation we obtain

$$0 = \int d\psi d\bar{\psi}\, d\chi_1 d\bar{\chi}_1 d\chi_2 d\bar{\chi}_2 \exp\left[S + \Delta S\right]$$

$$\Delta S = \int d^4x \; -e(\bar{\psi}\gamma^\mu\gamma_5\psi - \frac{1}{2}\bar{\chi}_1\gamma^\mu\gamma_5\chi_1 - \frac{1}{2}\bar{\chi}_2\gamma^\mu\gamma_5\chi_2)\partial_\mu\alpha \tag{24}$$

$$+ e\alpha M_1\bar{\chi}_1\gamma_5\chi_1 + e\alpha M_2\bar{\chi}_2\gamma_5\chi_2$$

Integrating over χ_i we obtain the product of the inverses of the determinants of the kinetic operators for χ_1 and χ_2 (which are complex commuting spinors). One finds, expanding to first order α

$$\det{}^{-1}(A_1+B_1)\det{}^{-1}(A_2+B_2) = \prod_{i=1}^{2} (\det A_i)^{-1}[1-\mathrm{Tr}A_i^{-1}B_i]$$

$$A_i = -e\gamma^\mu(D_\mu + iA\gamma_5)-eM_i + \frac{e}{2}\gamma^\mu\gamma_5\partial_\mu\alpha$$

$$B_i = \alpha e M_i \gamma_5 \tag{25}$$

Re-exponentiating $(\det A)^{-1}$, one obtains, to first order in α, and dropping the axial vector field

$$\langle -e\bar{\psi}\gamma^\mu\gamma_5\psi + \frac{e}{2}\bar{\chi}_1\gamma^\mu\gamma_5\chi_1 + \frac{e}{2}\bar{\chi}_2\gamma^\mu\gamma_5\chi_2\rangle\partial_\mu\alpha$$

$$= \mathrm{Tr}(A_1^{-1}B_1 + A_2^{-1}B_2) = \sum_{i=1}^{2} \mathrm{Tr}\,\frac{-1}{(\not{D}+M_i)}(\alpha M_i\gamma_5)$$

$$= \sum_{i=1}^{2}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\,e^{-ikx}(-\alpha M_i\gamma_5)\frac{1}{(\not{D}+M_i)(\not{D}-M_i)}(\not{D}-M_i)e^{ikx}$$

$$= \sum_{i=1}^{2}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}(-\alpha M_i\gamma_5)\frac{1}{(\not{D}+i\not{k})^2 - M_i^2}(\not{D}+i\not{k}-M_i) \tag{26}$$

since the denominator contains an even number of Dirac matrices, we can drop the $\not{D} + iK$ in the numerator, and find

$$\sum_{i=1}^{2}\mathrm{Tr}\,A_i^{-1}B_i =$$

$$\sum_{i=1}^{2}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}(\alpha M_i^2\gamma_5)\frac{1}{-(k^2+M_i^2) + 2ik\cdot D + D^2 + \frac{1}{4}R}$$

$$= \sum_{i=1}^{2} \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr} \, \frac{(-\alpha M_i^2 \gamma_5)}{(k^2+M_i^2)} [1 + (2ik\cdot D + D^2 + \frac{1}{4} R)/(k^2+M_i^2)$$

$$+ \frac{(2ik\cdot D)^2 + (D^2 + \frac{1}{4} R)^2}{(k^2+ M_i^2)^2} + \{(2ik\cdot D)(2ik\cdot D)(D^2 + \frac{1}{4} R)$$

$$+ (2ik\cdot D)(D^2 + \frac{1}{4} R)(2ik\cdot D) + (D^2 + \frac{1}{4} R)(2ik\cdot D)(2ik\cdot D)\}/(k^2+M_i^2)^3$$

$$+ (2ik\cdot D)^4/(k^2 + M_i^2)^4] \tag{27}$$

The terms with ($\mathrm{tr}\ \gamma_5 D^2$) are logarithmically divergent, and the spinor trace does not vanish a priori. However, assuming that $\Sigma M_i^2 = 0$, these terms are regulated. This is the motivation for using two regulators. The result resembles our previous result for the spin 1/2 case in many respects; however there are some differences: the first wo terms with two D's no longer cancel but add up to $[-D^2 M^4_i]/(k^2 + M_i^2)^3$.

Using

$$\int \frac{d^4k(k^2)^m}{(k^2+ M^2)^n} = \pi^2 (M^2)^{2+m-n} \frac{\Gamma(n-m-2)\Gamma(m+2)}{\Gamma(n)} \tag{28}$$

we obtain

$$\sum_{i=1}^{2} \mathrm{Tr} A_i^{-1} B_i = \sum_{i=1}^{2} \mathrm{tr}(-\frac{\alpha}{32\pi^2} \gamma_5)[(M_i^2 D^2 + D^2 D^2 + \frac{1}{2} RD^2)$$

$$- \frac{2}{3} (2D^2D^2 + \frac{3}{4} RD^2 + D_\mu D^2 D^\mu)$$

$$+ \frac{1}{3} (D^2 D^2 + D_\mu D_\nu D^\mu D^\nu + D_\mu D^2 D^\mu)] \tag{29}$$

As before we have dropped terms with $D_\mu R = \partial_\mu R$ since they do not contain enough Dirac matrices. The terms with RD^2 cancel again, (the numerical coefficients coming from the k-integral are essential for this), and the R-independent terms yield again a double commutator.

$$\sum_{i=1}^{2} \mathrm{Tr} A_i^{-1} B_i = \sum_{i=1}^{2} \mathrm{tr}\left(\frac{\alpha}{32\pi^2}\gamma_5\right)\{M_i^2 D^2 + \frac{1}{3}[D_\mu,D_\nu][D^\mu,D^\nu]\} \tag{30}$$

If we choose the regulator measses such that $M_1^2 + M_2^2 = 0$, then the M^2D^2 terms cancel, and one obtains

$$\sum_{i=1}^{2} \mathrm{Tr}\, A_i^{-1} B_i = \mathrm{tr}\left(\frac{\alpha}{32\pi^2}\gamma_5\right)\frac{1}{3}\,[D_\mu,D_\nu][D^\mu,D^\nu] \tag{31}$$

which is the same result as obtained from Fujikawa's method.

5. THE AXIAL ANOMALY IN N=1 CONFORMAL SUPERGRAVITY

In this section we use Fujikawa's method to compute the axial anomaly in simple (N=1) conformal supergravity. This is the first time that this method is used to obtain a new result. Of course, the method has also been used extensively to reobtain in elegant ways results which were previously obtained by laborious techniques such as the Feynman graph computations by Bardeen and Adler-Rosenberg. In our case, the Feynman graph calculations would have been very cumbersome, while Fujikawa's method is quite simple, despite the fact that one is dealing with a higher-derivative theory, due to a lemma (see below) according to which one may replace the regulator $\not{D}^3$ by $\not{D}$ in certain cases.

We begin by using a result derived by Fradkin and Tseytlin[5], who cast that part of the N=1 conformal supergravity action which contributes to one-gravitino-loop graphs, in a simple form. It reads

$$\mathcal{L} = -\bar{\psi}_\rho \not{D}^3 \psi_\rho - \frac{2}{3}\bar{\phi}\not{D}\phi + \frac{1}{2}\bar{\chi}\not{D}^3\chi \tag{32}$$

where the gravitationally covariant derivative D_μ contains both a spin-connection and a Christoffel part, and where further

$$\phi \equiv D^{\mu}\psi_{\mu} + \frac{1}{2}\not{D}\chi \ , \ \chi \equiv \gamma^{\mu}\psi_{\mu} \tag{33}$$

We have put the chiral gauge field A_{μ} to zero. Moreover, we have assumed that $R_{\mu\nu} = 0$, which is necessary in order that the gravitino action by itself be invariant under the ordinary and conformal supersymmetry transformations, which read

$$\delta_Q \psi_{\mu} = D_{\mu}\epsilon_Q \ ; \ \delta_S \psi_{\mu} = \gamma_{\mu}\epsilon_S \tag{34}$$

The easiest way to see that the gravitino part is separately invariant, is to note that its variation must cancel the variation of the Weyl action $R_{\nu\mu}{}^2 - \frac{1}{3}R^2$. Since the latter variation vanishes when $R_{\mu\nu} = 0$, so does the former.

We will cancel the last two terms in the gravitino action by adding the following gauge fixing terms

$$\mathcal{L}\,(\mathrm{fix}) = \frac{2}{3}\,\bar{\phi}\not{D}\phi - \frac{1}{2}\,\bar{\chi}\not{D}^3\chi \tag{35}$$

Thus we have two gauge fixing terms, namely $F^{\alpha} = (\phi,\chi)$, which will yield the usual Faddeev-Popov ghosts, while we also must take into account that the normalization determinates of $\not{D}$ and $\not{D}^3$ will give rise to Nielson-Kallosh ghosts. The Faddeev-Popov ghost action is obtained by varying F^{α} with respect to the (Q,S) supersymmetry transformations with parameters $\xi^{\alpha} = (\epsilon_Q,\epsilon_S)$, and sandwiching the result with commuting ghosts and antighosts. This leads to

$$\mathcal{L}\,(\mathrm{FP\ ghosts}) = \bar{C}_Q\left(\frac{3}{2}\,\Box\, C_Q + 3DC_S\right) + \bar{C}_S\left(\not{D}C_Q + 4C_S\right) \tag{36}$$

The exponentiation of the gauge fixing terms in the Dirac delta functions in the path-integral requires as normalization factors $(\det \not{D})^{-1/2}$ and $(\det \not{D})^{-3/2}$, respectively, since ϕ and χ are anti-commuting Majorana spinors. These determinants cannot be exponentiated as they stand, because they would require a <u>commuting</u> Majorana spinor whose Dirac action is a total derivative. The resolution is by now well-known[8]: one replaces $\not{D}^{-1/2}$ by $\not{D}^{-1}\not{D}^{1/2}$ and exponentiates by introducing a complex commuting ghost F and a Majorana anticommuting ghost f. Similar remarks apply to $(\det \not{D})^{-3/2}$ with ghosts G and g. In this way one obtains the following ghosts

$$\mathcal{L}(\text{NK ghosts}) = -\bar{F}\not{D}F - \bar{f}\not{D}f - \bar{G}\not{D}^3 G - \bar{g}\not{D}^3 g. \tag{37}$$

We must now determine the chiral weights of all these ghosts. For the Faddeev-Popov ghosts we follow ref.[7], and consider interaction terms like

$$\bar{C}_Q(D^{\mu}\partial_{\rho}\psi_{\mu}C^{\rho}_{g.c.}) + \bar{C}_S(\gamma^{\mu}\partial_{\rho}\psi_{\rho}C^{\rho}_{g.c.}) + \ldots \tag{38}$$

where $C^{\rho}_{g.c.}$ is the general coordinate ghost. Requiring chiral invariance of the complete Faddeev-Popov action yields the following chiral weights.

$$w(\psi_{\mu}) = +1,\ w(\bar{\psi}_{\mu}) = +1,\ w(\bar{C}_Q) = -1,\ w(\bar{C}_S) = +1$$

$$w(C_Q) = +1,\ w(C_S) = -1. \tag{39}$$

For the determination of the chiral weights of the NK ghosts we follow ref.[9] Namely we require that the part of the chiral current corresponding to the gauge-fixing term is cancelled by the part of the chiral current due to the NK ghosts. This is because in the unweighted gauge neither gauge fixing terms nor NK term are present in the quantum action. This argument shows at once that the NK Q-ghosts and the NK S-ghosts have opposite chiral weights.

We can simplify the quantum action by intergrating in the path integral over $\bar{C}_s$. This yields a Dirac-delta function $\delta(4C_s + \not{D}C_Q)$. which replaces C_s by $-\frac{1}{4}\not{D}C_Q$ after integration over C_s. Hence we finally obtain

$$\mathcal{L}(\text{quantum}) = -\bar{\psi}_{\rho}\not{D}^3\psi_{\rho} + \frac{3}{4}\bar{C}_Q \Box C_Q + \mathcal{L}(\text{NK ghosts}) \tag{40}$$

We now turn to the computation of the chiral anomaly associated with the global chiral invariance of $\mathcal{L}$(quantum). We must evaluate according to Fujikawa's method, the trace of the matrix $\gamma_5 w$ for all fields, where w is the chiral weight of the field considered. Moreover, in the computation one must regularize the ill-defined sum over spacetime points by the same operator as appears in the action. The justification of this procedure can, in our opinion, best be seen by comparing this Fujikasw computation with a Pauli-Villars computation in which the Jacobian is exactly unity but where now the anomaly resides in Pauli-Villars fields (see before). Hence, we use as regulators the operators exp R^2 where R is given by $(\not{D}/M)^3$ for ψ_ρ .

We now argue that the sum of the ghost contributions to the axial anomaly cancels. For the FP ghosts this follows easily from the fact that $\bar{C}_Q$ and C_Q have opposite chiral weights. This means that their contributions cancel in the Jacobian. For the NK ghosts, we make use of a general property of both the Fujikawa and the Pauli-Villars methods[1), namely that any regulator f(R) with f(0) = 1 and vanishing sufficiently fast at infinity , will give the same result. Hence, instead of $\not{D}^3$ operators we can take simply the $\not{D}$ operator as regulator. Since both F and G, and also both f and g, have the same regulators but opposite chiral weights, their contributions cancel.

All that remains to be computed is the anomaly due to ψ_ρ with regulator $\not{D}^3$, or rather, invoking the arguments presented above, with regulator $\not{D}$. This computation was already performed in section (3). Hence we conclude

chiral anomaly of N=1 conformal supergravity theory = (-20)A

where A is the chiral anomaly of a real anticommuting electron.

We conclude this section with a few comments.

1. The chiral weights can also be determined by keeping the chiral gauge field A_μ in the quantum action, and defining the chiral current by the coupling terms to A_μ . As shown by Fradkin and Tseytlin, A_μ appears as

$$-\bar{\psi}_\rho \not{D}\not{D}^+\not{D}\psi_\rho - \frac{2}{3}\bar{\phi}\not{D}\phi + \frac{1}{2}\bar{\chi}\not{D}^+\not{D}\not{D}^+\chi \tag{41}$$

where

$$\not{D} = \gamma^\mu(D_\mu - \frac{3i}{4}A_\mu\gamma_5) \text{ and } \not{D}^+ = \gamma^\mu(D_\mu + \frac{3i}{4}A_\mu\gamma_5). \tag{42}$$

Clearly, $\mathcal{L}$(class) is locally chiral invariant. From the exponentiation of $\not{D}^{-1/2}$, it follows that the commuting Majorana spinor (which should really be replaced by F and f as we explained) has the same weight as the gravitino. The chiral weight of G and g follows most easily by writing

$$\det(\not{D}^+\not{D}\not{D}^+)^{-1/2} = (\det\not{D}^+)^{-1/2}(\det\not{D})^{-1/2}(\det\not{D}^+)^{-1/2} \tag{43}$$

and observing that after exponentiation one would have three chiral currents from three commuting Majorana NK ghosts with weights -1, +1 and -1 respectively. Hence, the chiral contributions of the NK ghosts indeed cancel. The FP ghost action contains the following terms linear in A

$$\bar{C}_Q[\frac{3}{4}D^\mu D_\mu - \frac{1}{4}D_\mu\gamma^{\mu\nu}D_\nu]C_Q \tag{44}$$

This yields a chiral current

$$j_\mu = \frac{3}{4}\bar{C}_Q\overleftrightarrow{\partial}_\mu\gamma_5 C_Q - \frac{1}{4}\partial_\nu(\bar{C}_Q\gamma_{\mu\nu}\gamma_5 C_Q) \tag{45}$$

The first type of current has no anomaly while the second current is identically conserved.

2. For higher N models, spin 1/2 will contribute, in addition to N gravitinos. In particular, for N=4, one expects finiteness, since Fradkin and Tseytlin showed that the β function vanishes in this model. Actually, the local guage group is SU(4)in this model, not U(4), and although there is a composite local U(1) gauge invariance, it is not part of the expected anomaly multiplet. Hence, in the N=4 model vanishing of the chiral anomaly is replaced by absence of a chiral current; a rather trivial solution.

3. The N=1 conformal supergravity theory by itslef has anomalies in its coupling of the axial vector field, and thus, this theory is inconsistent. However, one can cancel these anomalies by coupling conformal matter.

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Superspace Path Integral Measure of the $N=1$ Spinning String

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Using the superspace formulation of the $N=1$ spinning string, we obtain a path integral measure which is free from world-sheet general-coordinate as well as Q-supersymmetry anomalies. Using this measure the conformal anomaly is explicitly calculated by extending Fujikawa's method to superspace. A complete solution of the 2-dimensional supergravity constraints is given.

1. Introduction

Fujikawa [1] has emphasized the importance of the functional measure for the understanding of anomalies: Different choices of the measure lead to different anomalies. For example, in Yang–Mills theories the requirement of gauge invariance determines a measure for the fermions that gives the correct chiral anomaly. Similarly, for systems coupled to gravity, coordinate invariance determines a measure that gives the correct trace anomaly. For a scalar field ϕ one finds that the correct measure, in any dimension d, is $D(g^{1/4}\phi)$ (where g is the determinant of the metric), whereas for a covariant vector field A_m with curved index the measure is $D(g^{d-2/4d}A_m)$. If the measure cannot be chosen to be invariant under all symmetries of the action, one has anomalies in each of the violated symmetries. One computes the anomaly by carefully defining (i. e., regulating) and evaluating the Jacobian determinant of the transformation. The measures above are found by requiring "naive invariance," i. e., that the unregulated Jacobian for general coordinate transformations be a total derivative.

When the gravitational field itself is quantized, the natural symmetry one would like to impose on the measure is BRST invariance, which is the residual rigid symmetry that remains at the quantum level after the local classical invariance has been fixed and coordinate ghosts have been introduced. Since the coordinate ghosts and antighosts serve the purpose of removing the unphysical degrees of freedom of the graviton at the quantum level, their measure should be treated simultaneously with that of the graviton. Moreover, since antighosts never have an antighost field in their BRST transformation rule, their contribution to the linear term in the Jacobian vanishes identically. Requiring that the product of the measures of the

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graviton and the ghost be BRST invariant, one should be able to deduce the correct choice of integration variables. However, technical problems in this approach [1, 2] lead to a modified prescription in which one treats the vielbein $e_m{}^a$ and the ghost C^m as covariant and contravariant vectors, respectively, and requires their measure to be invariant under general coordinate transformations. This prescription is known to yield correct results in all examples [1, 2].

In this article we extend these ideas to superspace. We find the supercoordinate invariant measure for scalar superfields ϕ is $D(\phi E^{-1/2})$ where $E = \text{sdet}\, E_A{}^M$ is the superdeterminant of the inverse vielbein. To determine the measure for the vielbein and ghost superfields, we follow the prescription above and require invariance under supercoordinate transformations for contravariant supervectors. This procedure leads to two surprising results: the measures for the vector and spinor components of the supervectors have *different* powers of E, and these powers are *not* uniquely determined by supercoordinate invariance. We fix the measure by *assuming* that for the superghosts with flat indices the measure is the same as that for scalars, and then transforming to curved indices [12]. Having found the correct measure, we rederive the critical dimension of the $N = 1$ spinning string. (Using different methods, Martinec [3] gave the first superspace derivation of the critical dimension). This concludes the body of the paper.

Several related issues are discussed in the appendices. In Appendix A we present our notation and conventions, and review $D = 2$ superspace. In Appendix B, we describe $D = 2$ superspace supergravity and give a complete solution of the Bianchi identities and the constraints in an arbitrary gauge; solutions in restricted gauges have been given in [4, 5]. We also derive some of the results of Sections 2–4 in spinor notation. In Appendix C we identify the supervielbein components with x-space fields using a nontrivial extension of the gauge completion method [6]. Finally, in Appendix D we derive a lemma for the evaluation of regulated super traces.

2. The Superspace Quantum Action of the Spinning String

In superspace the dynamics of a spinning string can be described by matter superfields $X^i(x, \theta)$ $(i = 1, ..., d)$ coupled to the 2-dimensional supergravity multiplet $E_A{}^M(x, \theta)$ [5]. However, the 16 components of $E_A{}^M$ contain too many x-space fields, and these are eliminated by imposing constraints. Following [3], we choose the following set of constraints on the torsion tensor $T_{AB}{}^C$ (the notation is summarized in Appendix A).

$$T_{\alpha\beta}{}^c = 2i(\gamma^c)_{\alpha\beta} \tag{2.1}$$

$$T_{\alpha\beta}{}^\gamma = 0 \tag{2.2}$$

$$T_{ab}{}^c = 0. \tag{2.3}$$

The Bianchi identity

$$D_{(\alpha} T_{\beta\gamma)}{}^{d} + T_{(\alpha\beta|}{}^{E} T_{E|\gamma)}{}^{d} = 0$$

implies a further constraint

$$T_{\alpha b}{}^{c} = 0. \tag{2.4}$$

In Appendix B a complete solution of these constraints in terms of unconstrained superfields as well as a solution of the Bianchi identities is given. For the purposes of the present paper we do not need this explicit solution. Instead we reproduce here the argument given by Martinec [3]. Just as in four dimensions, (2.3) can be used to express the bosonic connection ϕ_a in terms of the vielbein, while (2.1), (2.2), and (2.4) determine the bosonic vielbein $E_a{}^M$ in terms of the fermionic vielbein $E_\alpha{}^M$. The constraint (2.2) not only expresses the fermionic connection ϕ_α in terms of the vielbein, it also provides two more constraints on the fermionic vielbein $E_\alpha{}^M$. Therefore we have only six independent components of $E_\alpha{}^M$ left. If one fixes the gauge for the 4 supercoordinate transformations and the 1 local Lorentz transformation, one is left with only one superfield degree of freedom, the conformal factor ψ. This corresponds to the x-space conformal gauge, where one has only the trace of the graviton, the γ trace of the gravitino, and a single auxiliary field left. By direct computation, or as explained in Appendix B, one can check that the constraints (2.1)–(2.4) are satisfied by

$$E_\alpha = e^{\psi} D_\alpha, \qquad E_a = e^{2\psi} \partial_a + i e^{2\psi} \gamma_a{}^{\beta\gamma} (D_\gamma \psi) D_\beta. \tag{2.5}$$

(We have not distinguished flat and curved indices on the rigid superspace derivatives, i. e., $D_A \equiv \delta_A{}^M D_M$. See Appendix A for further details of notation). Locally, *any* vielbein can be obtained by applying a gauge (Lorentz + supercoordinate) transformation to (2.5).

In Polyakov's approach to string theory [7], one treats the vielbein $E_A{}^M$ as a dynamical variable and integrates over it in the path integral. One therefore fixes the gauge to factorize out the volume of the gauge transformations. We make the following five gauge choices:

$$\begin{aligned} &E_\alpha{}^m = 0 \\ &E_1{}^1 = E_2{}^2 = e^{\psi} \end{aligned} \tag{2.6}$$

(1 and 2 are fermionic indices and are called + and − in Appendix B) to fix the 4 supercoordinate transformations and the 1 local Lorentz transformation. Given these gauge choices, the torsion constraints imply that the vielbein is of the conformal form (2.5). To see this, we have to show that $E_1{}^2 = E_2{}^1 = 0$, since the fermionic vielbein is then completely specified (see also Appendix B). The bosonic vielbein is

then of the form in (2.5) since the bosonic vielbein is solved in terms of fermionic vielbein through the torsion constraints. To see that $E_1{}^2 = E_2{}^1 = 0$, let us consider

$$\begin{aligned}\{\nabla_\alpha, \nabla_\beta\} &= T_{\alpha\beta}{}^C \nabla_C + R_{\alpha\beta} M \\ &= 2i(\gamma^c)_{\alpha\beta} \nabla_c + R_{\alpha\beta} M.\end{aligned} \tag{2.7}$$

Since $(\gamma^c)_{\alpha\beta}$ is diagonal (see Appendix A), we have

$$\{\nabla_1, \nabla_2\} = R_{12} M. \tag{2.8}$$

In the gauge of (2.6), we have

$$\begin{aligned}\nabla_1 &= e^\psi D_1 + E_1{}^2 D_2 + \phi_1 M \\ \nabla_2 &= e^\psi D_2 + E_2{}^1 D_1 + \phi_2 M.\end{aligned} \tag{2.9}$$

On the left-hand side of (2.8), we collect terms involving space-time derivatives:

$$\begin{aligned}&e^\psi E_2{}^1 \{D_1, D_1\} + e^\psi E_1{}^2 \{D_2, D_2\} \\ &\quad = -2ie^\psi E_2{}^1(\partial_0 + \partial_1) - 2ie^\psi E_1{}^2(\partial_0 - \partial_1).\end{aligned} \tag{2.10}$$

Since the right-hand side of (2.8) does not contain any terms with space time derivatives, we conclude that

$$E_2{}^1 = E_1{}^2 = 0. \tag{2.11}$$

This shows that in the gauge (2.6) the vielbein is of the conformal form (2.5).

In the following we work in the unweighted gauge of (2.6), i. e., we insert explicit δ-functions in the path integral. The next step is to construct the Faddev–Popov ghost action. To do this we have to consider the gauge variations of (2.6). In general, we define

$$\delta\nabla_A = \delta E_A{}^M D_M + \delta\phi_A M = [E_A{}^M D_M + \phi_A M, K^N D_N + \Lambda \cdot M] \tag{2.12}$$

where K^N and Λ are the gauge parameters for the supercoordinate transformations and the local Lorentz transformations, respectively. For later use, we will split the gauge fixing term $E_\alpha{}^m = 0$ into its γ trace and γ traceless parts:

$$\begin{aligned}(\bar\gamma_m)^{\mu\beta} E_\beta{}^m &= 0 \\ \hat E_\alpha{}^m = E_\alpha{}^m - \tfrac{1}{2}(\bar\gamma^m \bar\gamma_n)_\alpha{}^\beta E_\beta{}^n &= 0\end{aligned} \tag{2.13}$$

where $\bar\gamma$ represents constant Dirac matrices regardless of its indices.

The variations of these gauge fixing terms are then given by

$$\begin{aligned}\delta(\bar\gamma_m{}^{\mu\beta} E_\beta{}^m) &= e^\psi \bar\gamma_m{}^{\mu\beta} D_\beta K^m + 4ie^\psi K^\mu \\ \delta(\hat E_\alpha{}^m) &= \tfrac{1}{2} e^\psi (\bar\gamma_n \bar\gamma^m)_\alpha{}^\beta D_\beta K^n \\ \delta(E_1{}^1 - E_2{}^2) &= e^\psi (D_1 K^1 - D_2 K^2) + e^\psi \Lambda.\end{aligned} \tag{2.14}$$

The Fadeev–Popov ghost action is then obtained by replacing the gauge parameters by the corresponding ghosts and multiplying the variations with their associated antighosts:

$$\begin{aligned}\mathscr{L}_{\text{ghost}} = {} & \tfrac{1}{2}\bar{C}_m{}^{\alpha} e^{\psi} (\bar{\gamma}_n \bar{\gamma}^m)_{\alpha}{}^{\beta} D_{\beta} C^n \\ & + \bar{C}_{\mu} e^{\psi} (\bar{\gamma}_m{}^{\mu\beta} D_{\beta} C^m + 4iC^{\mu}) \\ & + \bar{C} e^{\psi} (D_1 C^1 - D_2 C^2 + C).\end{aligned} \tag{2.15}$$

The antighost $\bar{C}_m^{\alpha}$ formally has 4 components; however, 2 of them, the γ-trace part, drop from the action (2.15) due to the 2-dimensional identity $\gamma_m \gamma_n \gamma^m = 0$. In light cone coordinates (see Appendix B) one can define an antighost which has only the relevant two components, but for our later calculations we find it easier to keep the ghost action in this form and remove the redundency later.

3. The Functional Measure in Superspace

Having determined the quantum action in the last section, we now turn to the problem of identifying the proper functional measure in the path integral. Choosing different functional measures will in general lead to different kinds of anomalies [1]. In the bosonic string, one fixes the functional measure by requiring that no world-sheet reparametrization anomaly be present. Here, in the superspace approach to the spinning string, we should choose a new functional measure that is invariant under supercoordinate transformations; this guarantees the absence of the Q-supersymmetry anomalies as well. The method we are going to use is very similar to the bosonic case; however, we find the surprising result that in $d=2$ the invariance under supercoordinate transformations does not fix the functional measure for a supervector field uniquely. Before discussing this problem, let us first look at the proper measure for a scalar superfield S. Under supercoordinate transformations a scalar superfield transforms as

$$\delta S = [S, K^M D_M] = -K^M D_M S. \tag{3.1}$$

We assume that the proper integration variable is $\tilde{S} = SE^k$, where $E = \operatorname{sdet} E_A{}^M$. (Note that in the literature, usage varies and E is often defined as sdet $E_M{}^A$.) The transformation law of $E_A{}^M$ is defined by

$$\delta E_A{}^M D_M = [E_A{}^M D_M, K^N D_N] \tag{3.2}$$

(cf. 2.12). Therefore

$$\delta E_A{}^M = E_A{}^N D_N K^M - K^N D_N E_A{}^M - 2iE_A{}^{\mu} K^{\nu} \bar{\gamma}^m_{\mu\nu} \delta_m{}^M \tag{3.3}$$

and

$$\delta E = E(D_M K^M - E_M{}^A K^N D_N E_A{}^M)(-)^M \tag{3.4}$$

Note that the last term in (3.3), coming from the anticommutator $\{D_\alpha, D_\beta\}$, cancels in (3.4). Hence we obtain the following transformation of $\tilde{S}$:

$$\delta\tilde{S} = -K^M D_M \tilde{S} + k\tilde{S} D_M K^M(-)^M. \tag{3.5}$$

The Jacobian of this transformation is $\exp \operatorname{str}(\partial\delta\tilde{S}/\partial\tilde{S})$, where the supertrace can be defined by using a complete orthonormal set of superfields ϕ^i for $\tilde{S}$ as follows:

$$\operatorname{str}\frac{\partial\delta\tilde{S}}{\partial\tilde{S}} = \sum_i \int d^2x\, d^2\theta\, \phi^i[-K^M D_M + k(D_M K^M)(-)^M]\,\phi^i. \tag{3.6}$$

By partially integrating the second term in (3.6), we obtain, up to a total derivative:

$$\operatorname{str}\frac{\partial\delta\tilde{S}}{\partial\tilde{S}} = \sum_i \int d^2x\, d^2\theta\, \phi^i(-1-2k)\, K^M D_M \phi^i. \tag{3.7}$$

The condition for a unit Jacobian is therefore

$$k = -\tfrac{1}{2} \qquad \text{or} \qquad \tilde{S} = S(\operatorname{sdet} E_A{}^M)^{-1/2}. \tag{3.8}$$

Recall that in the x-space case one has

$$\tilde{S} = S(\det e_a{}^m)^{-1/2}$$

so the only difference is that the vielbein is replaced by the supervielbein. Since under BRST transformations the antighost always transforms into the auxiliary field, its Jacobian trivially equals unity, just as in x space. However, subtle differences arise when one looks at the functional measure of the vielbein and the ghosts. The issue is even rather obscure in x space. In principle, the functional measure should follow from the requirement that the Jacobian of the BRST transformation be unity. In x space, the measure of the ghost and vielbein has been derived by letting these fields transform under BRST transformations as contravariant vectors. This procedure has not yet been justified, although whenever it as been used to compute anomalies it has given the correct results. Therefore we adopt this procedure in superspace. In ordinary space-time, the functional measure of the general coordinate ghost C^m is given by

$$\tilde{C}^m = C^m(\det e_a{}^m)^{-(d+2)/2d} \tag{3.9}$$

where d is the dimension of space-time) and the measure of the supersymmetry ghost C^α is the same as that of a scalar. Two questions immediately arise: What is the equivalent of d in superspace and will the functional measure be the same for C^m and C^α in superspace? To answer these questions, let us follow the rule stated above and investigate the functional measure of a supercontravariant vector V^M. Under supercoordinate transformations we have

$$\delta V^M = V^N D_N K^M - K^N D_N V^M - 2iV^\mu K^\nu \bar{\gamma}^m_{\mu\nu}\delta_m{}^M. \tag{3.10}$$

Let us first assume $\tilde{V}^M = E^k V^M$. The Jacobian is

$$J = \exp \operatorname{str} \frac{\partial \delta \tilde{V}^M}{\partial \tilde{V}^N} = \exp \frac{\partial \delta \tilde{V}^M}{\partial \tilde{V}^M} (-)^M. \tag{3.11}$$

The last term in (3.10) does not contribute to the supertrace so that we drop it in the following discussion. We then have

$$\delta \tilde{V}^M = \tilde{V}^N D_N K^M - K^N D_N \tilde{V}^M + k \tilde{V}^M (D_N K^N)(-)^N. \tag{3.12}$$

In $d=2$, the number of bosonic and (Majorana) fermionic components are both equal to two, so in the supertrace $(\partial \delta \tilde{V}^M / \partial \tilde{V}^M)(-)^M$, the last two terms of (3.12) always cancel separately, leaving only a non-vanishing contribution from the first term. In this case the Jacobian would never be unity. The only way out is to take k different for V^m and V^μ. Denoting these numbers by k_m and k_μ, respectively, we obtain ($\tilde{V}^m \equiv V^m E^{k_m}$, $\tilde{V}^\mu \equiv V^\mu E^{k_\mu}$),

$$\frac{\partial \delta \tilde{V}^m}{\partial \tilde{V}^m} = \sum_i \int d^2x\, d^2\theta\, \phi^i [\partial_m K^m - 2K^m \partial_m - 2K^\mu D_\mu + 2k_m(\partial_m K^m - D_\mu K^\mu)]\, \phi^i \tag{3.13}$$

$$\frac{\partial \delta \tilde{V}^\mu}{\partial \tilde{V}^\mu} = \sum_i \int d^2x\, d^2\theta\, \phi^i [(D_\mu K^\mu) - 2K^m \partial_m - 2K^\mu D_\mu + 2k_\mu(\partial_m K^m - D_\mu K^\mu)]\, \phi^i. \tag{3.14}$$

The factors of 2 are due to the spinor or vector traces. Requiring that $(\partial \delta \tilde{V}^M / \partial \tilde{V}^M)(-)^M \equiv \partial \delta \tilde{V}^m / \partial \tilde{V}^m - \partial \delta \tilde{V}^\mu / \partial \tilde{V}^\mu$ be a total derivative gives

$$k_m - k_\mu = -\tfrac{1}{2}. \tag{3.15}$$

So we see that in $d=2$, the condition of a unit Jacobian does not fix k_m and k_μ uniquely. We *assume* that the measure for ghosts with flat indices is the same as that for scalars. Hence $\tilde{C}^A = C^A e^{-\psi}$. Transforming to curved indices we find $\tilde{C}^a = C^m e^{-3\psi}$ and $\tilde{C}^\alpha = C^\mu e^{-2\psi}$ [12]. Hence,

$$k_m = -\tfrac{3}{2} \qquad \text{and} \qquad k_\mu = -1. \tag{3.16}$$

These values satisfy (3.15). We thus obtain the following weights for the vielbein and ghosts

$$\begin{aligned} \tilde{E}_A{}^m &= e^{-3\psi} E_A{}^m, & \tilde{E}_A{}^\mu &= e^{-2\psi} E_A{}^\mu, \\ \tilde{C}^m &= e^{-3\psi} C^m, & \tilde{C}^\mu &= e^{-2\psi} C^\mu, \end{aligned} \tag{3.17}$$

where we have used that $E=e^{2\psi}$ from (2.5). The ghost weights are again those of a contravariant vector. At this point we see that the should actually have used

$$\tilde{E}_\alpha{}^m=0 \qquad \text{and} \qquad \tilde{E}_1{}^1-\tilde{E}_2{}^2=0 \tag{3.18}$$

as gauge fixing terms instead of (2.6). Taking this fact into account and expressing each field in terms of twiddled ones, we obtain the partition function for the spinning string:

$$Z=\int d\tilde{E}_A{}^M\, d\tilde{X}^i\, d\tilde{\bar{C}}_m{}^\alpha\, d\tilde{\bar{C}}^\mu\, d\tilde{\bar{C}}\, d\tilde{C}^m\, d\tilde{C}^\mu\, d\tilde{C}\, e^I$$

where

$$\begin{aligned} I=\tfrac{1}{4}\int d^2x\, d^2\theta\, [&D^\alpha(e^\psi \tilde{X}^i)\, D_\alpha(e^\psi \tilde{X}^i) \\ &+\tilde{\bar{C}}_m{}^\alpha e^{-2\psi}(\bar{\gamma}_n \gamma^m)_\alpha{}^\beta\, D_\beta(e^{3\psi}\tilde{C}^n) \\ &+\tilde{\bar{C}}_\mu(e^{-2\psi}\bar{\gamma}_m{}^{\mu\beta} D_\beta(e^{3\psi}\tilde{C}^m)+4i\tilde{C}^\mu) \\ &+\tilde{\bar{C}}e^{-\psi}(D_1(e^{2\psi}\tilde{C}^1)-D_2(e^{2\psi}\tilde{C}^2)+e^\psi\tilde{C})] \end{aligned} \tag{3.19}$$

and

$$\begin{gathered} \tilde{E}_A{}^m=e^{-3\psi}E_A{}^m, \qquad \tilde{E}_A{}^\mu=e^{-2\psi}E_A{}^\mu, \\ \tilde{X}^i=e^{-\psi}X^i, \qquad \tilde{\bar{C}}_m{}^\alpha=\bar{C}_m{}^\alpha, \qquad \tilde{\bar{C}}_\mu=\bar{C}_\mu, \qquad \tilde{\bar{C}}=\bar{C}, \\ \tilde{C}^m=e^{-3\psi}C^m, \qquad \tilde{C}^\mu=e^{-2\psi}C^\mu, \qquad \tilde{C}=e^{-\psi}C. \end{gathered}$$

4. The Conformal Anomaly

In this section, we obtain the effective action of the conformal factor ψ after integrating over $\tilde{X}^i$, the ghosts, and the antighosts. We notice that after the rescaling

$$\begin{aligned} &\tilde{X}^i \to e^{-\psi}\tilde{X}^i \\ &\tilde{C}^m \to e^{-3\psi}\tilde{C}^m, \qquad \tilde{C}^\mu \to e^{-2\psi}\tilde{C}^\mu, \qquad \tilde{C}\to e^{-\psi}\tilde{C}, \\ &\tilde{\bar{C}}_m{}^\alpha \to e^{2\psi}\tilde{\bar{C}}_m{}^\alpha, \qquad \tilde{\bar{C}}_\mu \to e^{2\psi}\tilde{\bar{C}}_\mu, \qquad \tilde{\bar{C}}\to e^\psi\tilde{\bar{C}}, \end{aligned} \tag{4.1}$$

all these fields become decoupled from the conformal factor ψ, so the integrations just yield a ψ-independent constant. However, there will be non-trivial Jacobian factors coming from the change of integration variables (4.1), which have to be regularized carefully. The finite change of variables in (4.1) can be reached by performing a series of infinitesimal rescalings with scale parameter δt, and finally integrating over t from $t=0$ to $t=1$. In the intermediate stage, the ψ factor appearing in the action (3.19) is of the form

$$\psi_t = (1-t)\,\psi. \tag{4.2}$$

The Jacobian of the infinitesimal rescaling of each field is

$$J = \exp \operatorname{Tr}(q \cdot \psi\, \delta t) \tag{4.3}$$

where q is the product of the number components, statistics, and the weight of the rescaling as given in (4.1). (By including the statistics in q, we replace a supertrace by a trace.)

The trace has to be regularized for each field by the quadratic part of its action. Let us first discuss $\tilde{X}^i$. The kinetic operator of $\tilde{X}^i$ given by (3.19) is $Q_t = e^{\psi_t} D^\alpha D_\alpha e^{\psi_t}$. Since $Q_t{}^\dagger = Q_t$ there exists a complete set of orthonormal superfields

$$Q_t \phi_i = \lambda_i(t)\, \phi_i. \tag{4.4}$$

The trace in (4.3) can then be regularized by

$$\begin{aligned}\operatorname{Tr} q\psi\, \delta t &= \lim_{M\to\infty} \sum_i \int d^2x\, d^2\theta\, \phi^i q\psi\, \delta t \exp\left(-\frac{\lambda_i^2}{M^2}\right) \phi_i \\ &= \lim_{M\to\infty} \sum_i \int d^2x\, d^2\theta\, \phi_i q\psi\, \delta t \exp\left(-\frac{Q_t^2}{M^2}\right) \phi_i.\end{aligned} \tag{4.5}$$

Since a (super)trace is invariant under a change of basis, we may replace the ϕ_i in (4.5) by the superspace generalization of plane waves

$$e^{ikx + \theta^\alpha \chi_\alpha} \equiv e^{iZ\cdot K}. \tag{4.6}$$

The completeness relation

$$\int \frac{d^2k}{(2\pi)^2}\, d^2\chi\, e^{iZ\cdot K} e^{-iZ'\cdot K} = \delta^2(x-x')\,\delta^2(\theta-\theta')$$

can easily be verified. We therefore obtain

$$\begin{aligned}\operatorname{Tr} q\psi\, \delta t = \lim_{M\to\infty} &\int d^2x\, d^2\theta\, (q\cdot\psi\,\delta t) \int \frac{d^2k}{(2\pi)^2}\, d^2\chi \\ &\times e^{-ikx - \theta^\alpha\chi_\alpha} \exp -\left(\frac{e^{\psi_t} D^\alpha D_\alpha e^{\psi_t}}{M}\right)^2 e^{ikx + \theta^\alpha \chi_\alpha}.\end{aligned} \tag{4.7}$$

To make the integral well defined, we follow the usual procedure of analytically continuing from Minkowski to Euclidean space-time. Evaluating the square in

(4.7), we find a form such that a general lemma proven in Appendix D can be applied. Using this result and noting that $q = -d$ for $\tilde{X}^i$, we obtain

$$\int_0^1 \operatorname{Tr} q\psi\, dt = -\frac{d}{4\pi} - i\int_0^1 dt \int d^2x\, d^2\theta\, \psi D^2\psi_t$$

$$= \frac{d}{8\pi} - i\int d^2x\, d^2\theta\, (D^\alpha\psi)(D_\alpha\psi). \tag{4.8}$$

Let us now turn to the ghost rescalings. As a consequence of splitting the gauge fixing term $\tilde{E}^m_\alpha = 0$ into its γ trace and γ-traceless parts, our ghost action (3.19) is of triangular form. Therefore, in computations involving closed ghost loops the off-diagonal terms never contribute. We notice furthermore, that the C^μ and C ghosts are nonpropagating, so closed loops involving these ghosts will not contribute to the anomaly either. Therefore, the only ghost term we have to consider is

$$\tilde{\bar{C}}_m{}^\alpha e^{-2\psi}(\bar{\gamma}_n\bar{\gamma}^m)_\alpha{}^\beta D_\beta e^{3\psi}\tilde{C}^n. \tag{4.9}$$

The problem here is that the kinetic operator $e^{-2\psi}(\bar{\gamma}_n\bar{\gamma}^m)_\alpha{}^\beta D_\beta e^{3\psi}$ is not hermitian. The solution to this problem is well known [1, 8]. In general, if the ghost action is of the form $\bar{C}\mathbf{O}C$, where $\mathbf{O}$ is not a hermitian operator, we can consider the hermitian actions $C\mathbf{O}^\dagger\mathbf{O}C$ and $\bar{C}\mathbf{O}\mathbf{O}^\dagger\bar{C}$, and take $\mathbf{O}^\dagger\mathbf{O}$ and $\mathbf{O}\mathbf{O}^\dagger$ as hermitian operators to regularize the trace of the ghost and antighosts, respectively. In our case, we have for the ghost

$$\tilde{C}^m e^{3\psi}(\bar{\gamma}_p\bar{\gamma}_m)^{\alpha\beta} D_\beta e^{-4\psi}(\bar{\gamma}_n\bar{\gamma}^p)_\alpha{}^\gamma D_\gamma e^{3\psi}\tilde{C}^n$$

$$= -2\tilde{C}^m e^{3\psi} D_\alpha e^{-4\psi} D_\beta e^{3\psi}(\bar{\gamma}_m\bar{\gamma}_n)^{\alpha\beta}\tilde{C}^n. \tag{4.10}$$

The operator

$$Q_{mn}(t) = e^{3\psi_t} D_\alpha e^{-4\psi_t} D_\beta e^{3\psi_t}(\bar{\gamma}_m\bar{\gamma}_n)^{\alpha\beta} \tag{4.11}$$

is now hermitian and we can therefore use it to regularize the trace of the $\tilde{C}^m$ ghost in the same way as we did for $\tilde{X}^i$:

$$\operatorname{Tr} q\psi\,\delta t = \lim_{M\to\infty}\sum_i \operatorname{tr}\int d^2x\, d^2\theta\, \phi_i q\psi\,\delta t \exp\left(-\frac{Q_{mp}(t)\,Q^{pn}(t)}{M^2}\right)\phi_i \tag{4.12}$$

where the "tr" denotes trace in (mn) space, i.e., a summation over $m = n$.

For the antighost $\tilde{\bar{C}}^\alpha_m$ we have first to fix the gauge, since as mentioned Section 2, (4.9) has a local gauge invariance under

$$\delta\tilde{\bar{C}}_m{}^\alpha = \bar{\gamma}_m{}^\alpha{}_\beta\chi^\beta \tag{4.13}$$

where χ^β is a arbitrary Majorana spinor. We use these two degrees of gauge freedom to impose two conditions on the antighosts.

$$\tilde{\bar{C}}_m{}^1 = -\tilde{\bar{C}}_m{}^2 \equiv \tilde{\bar{C}}_m. \tag{4.14}$$

Since this is an algebraic gauge, there are no "ghosts for ghosts." So we obtain the antighost action from (3.19)

$$\begin{aligned}&\bar{\bar{C}}_m e^{-2\psi} D_\alpha e^{6\psi} D_\beta e^{-2\psi} \bar{\bar{C}}_n [(\bar{\gamma}_p)_{1\delta}(\bar{\gamma}^m)^{\delta\alpha} - (\bar{\gamma}_p)_{2\delta}(\bar{\gamma}^m)^{\delta\alpha}] \\ &\quad \times [(\bar{\gamma}^n)^{\beta\lambda}(\bar{\gamma}^p)_{1\lambda} - (\bar{\gamma}^n)^{\beta\lambda}(\bar{\gamma}^p)_{2\lambda}] \\ &\quad = -2\bar{\bar{C}}_m e^{-2\psi} D_\alpha e^{6\psi} D_\beta e^{-2\psi} (\bar{\gamma}^m)^{\alpha\delta}(\bar{\gamma}_5)_{\delta\lambda}(\bar{\gamma}^n)^{\lambda\beta}\, \bar{\bar{C}}_n \end{aligned} \tag{4.15}$$

where we have used the identity

$$(\bar{\gamma}_p)_{\alpha\beta}\, (\bar{\gamma}^p)_{\lambda\delta} = -\varepsilon_{\alpha\lambda}\varepsilon_{\beta\delta} + (\bar{\gamma}_5)_{\alpha\lambda}(\bar{\gamma}_5)_{\beta\delta}. \tag{4.16}$$

The operator

$$\bar{Q}_{mn}(t) = e^{-2\psi_t} D_\alpha e^{6\psi_t} D_\beta e^{-2\psi_t} (\bar{\gamma}_m \bar{\gamma}_5 \bar{\gamma}_n)^{\alpha\beta} \tag{4.17}$$

will then be used in the regulator

$$\exp\left[-\frac{\bar{Q}_{mp}(t)\, \bar{Q}^{pn}(t)}{M^2} \right]. \tag{4.18}$$

Working out $Q_{mp}(t)\, Q^{pn}(t)$ and $\bar{Q}_{mp}(t)\, \bar{Q}^{pn}(t)$ in (4.12) and (4.18) respectively one easily sees that they are the same in structure and differ only in $e^{-\psi_t}$ factors. Decomposing them into a sum of terms proportional $\delta_m{}^n$ and $\varepsilon_m{}^n$ respectively, we see that the $\varepsilon_m{}^n$ terms do not contribute. Hence the operator in (4.18) again reduces to a form suitable for the lemma in Appendix D. Taking into account that $q = 6$ for the ghost and $q = 4$ for the antighost, we obtain

$$\begin{aligned}\int_0^1 \operatorname{Tr} q\psi\, \delta t &= \frac{9}{2\pi} - i \int_0^1 dt \int d^2x\, d^2\theta\, \psi D^2 \psi_t \\ &= -\frac{9}{4\pi} \int d^2x\, d^2\theta\, (D^\alpha \psi)(D_\alpha \psi) \end{aligned} \tag{4.19}$$

for the ghost and

$$\begin{aligned}\int_0^1 \operatorname{Tr} q\psi\, \delta t &= \frac{-2}{\pi} \int_0^1 dt \int d^2x\, d^2\theta\, \psi D^2 \psi_t \\ &= \frac{1}{\pi} \int d^2x\, d^2\theta\, (D^\alpha \psi)(D_\alpha \psi) \end{aligned} \tag{4.20}$$

for the antighosts.

Summing over the contributions from $\tilde{X}^i$, the ghost, and the antighost, we obtain

$$\frac{d-10}{8\pi} - i \int d^2x\, d^2\theta\, (D^\alpha \psi)(D_\alpha \psi) \tag{4.21}$$

in agreement with the previous results [7, 3].

5. Conclusion

In a previous computation [2] of critical dimensions of spinning strings, a measure was used which was invariant under BRST-general coordinate transformations. For supersymmetric systems, it is natural to require that the measure also be invariant under BRST-local supersymmetry transformations. In principle, this problem could have been tackled directly in x space, but due to the many ambiguities, we instead solved this problem in superspace. We found the curious result that BRST-supercoordinate invariance did not completely fix the measure of the supercoordinate ghosts. This may indicate that the quantum action has a further symmetry such that by also requiring that this symmetry is free from anomalies, the measure would get completely fixed. We fixed the measure by requiring that ghosts with flat superindices have the same measure as scalars, and further that the measure of the vielbeins be the same as that of the ghosts with curved indices [12]. With this measure we then computed the critical dimension of the spinning string, and found the correct results $d=10$.

Our results now allow one to determine the supersymmetric measure in x space. In [2], it was shown that at each point in x space, the sum of the Jacobians for local supersymmetry variations of the various fields did cancel. This, however, is not sufficient, as is clear from the fact that one would not obtain the fermionic terms in the supersymmetric extension of the Liouville action. Rather, one should regularize the Jacobian for each field, and sum these regularized Jacobians.

Appendix A: Notation and Conventions

Our metric is $\eta_{ab}=(-\,+)$ for $a=0, 1$. Spinorial indices are raised or lowered by $\varepsilon_{\alpha\beta}=\varepsilon^{\alpha\beta}$, $\varepsilon_{12}=1$, according to

$$\chi^\alpha=\varepsilon^{\alpha\beta}\chi_\beta, \qquad \chi_\alpha=\chi^\beta\varepsilon_{\beta\alpha}.$$

Majorana spinors are defined by

$$\chi_\alpha{}^\dagger=\chi_\alpha, \qquad \chi^{\alpha\dagger}=\chi^\alpha.$$

We use a real representation for the γ matrices

$$(\gamma_0)_\alpha{}^\beta=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad (\gamma_1)_\alpha{}^\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad (\gamma_5)_\alpha{}^\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$(\gamma_0)_{\alpha\beta}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \qquad (\gamma_1)_{\alpha\beta}=\begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}.$$

In curved superspace, we use $\bar{\gamma}$ to denote these constant γ matrices regardless of the indices on $\bar{\gamma}$. The rigid covariant spinor derivatives of flat superspace satisfy

$$\{D_\mu, D_\nu\} = 2i\bar{\gamma}^m_{\mu\nu}\partial_m, \qquad D_\mu = \partial_\mu + i(\bar{\gamma}^m)_{\mu\nu}\theta^\nu\partial_m$$
$$D^2 = D^\mu D_\mu$$
$$D^2 D_\mu = -D_\mu D^2 = 2i\bar{\gamma}^m_{\mu\nu}D^\mu\partial_m$$
$$(D^2)^2 = -4\Box.$$

Note that D^2 is purely antihermitian, whereas D_μ is hermitian. To see this, note that from $\{\partial_\mu, \theta^\nu\} = \delta^\nu_\mu$ and $[\partial_m, x^n] = \delta^n_m$ it follows that $(\partial_\mu)^\dagger = \partial_\mu$ and $(\partial_m)^\dagger = -\partial_m$. We use $A = (a, \alpha)$ to denote tangent superspace indices and $M = (m, \mu)$ to denote curved superspace indices.

Vielbeins are defined by

$$E_A = E_A{}^M D_M$$

where D_M are the covariant derivative of flat superspace

$$D_M = (\partial_m, D_\mu)$$

and covariant derivatives of curved superspace are defined by

$$\nabla_A = E_A + \phi_A M$$

where

$$M_A B = \begin{pmatrix} \varepsilon_a{}^b & 0 \\ 0 & \frac{1}{2}(\gamma_5)_\alpha{}^\beta \end{pmatrix}$$

is the Lorentz generator. Note that the vielbein $E_A{}^M$ is not exactly equal to the vielbein $V_A{}^M$ used in the earlier literature of supergravity; rather, $V_A{}^M$ is expanded on a holonomic basis by $E_A = V_A{}^M\partial_M$, which implies $E_A{}^M = V_A{}^N \mathring{E}_N{}^B \delta_B{}^M$ where $\mathring{E}_B{}^M$ is the vielbein of flat superspace.

We further define torsion and curvature tensor by

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C\nabla_C + R_{AB}M.$$

Grassmanian integration is defined by

$$d^2\theta = i d\theta^1 d\theta^2 = \frac{-i}{2}\varepsilon^{\alpha\beta}\, d\theta_\alpha\, d\theta_\beta = \frac{i}{2}\, d\theta^\alpha\, d\theta_\alpha;$$

where $d^2\theta$ is real, and $\delta^2(\theta) = i\theta^1\theta^2$.

Because the 2-dimensional Lorentz group is $SO(1, 1)$, which has only 1-dimensional representations, it is very convenient to work in a basis of helicity eigenstates. Then a spinor index μ takes two values $(+, -)$, which represent the helicity $\pm\frac{1}{2}$ components, $\frac{1}{2}(\gamma_5\chi)_\pm = \pm\frac{1}{2}\chi_\pm$, and a vector index m takes the values $(++, --)$, which represent the helicity ± 1 components and are equivalent to light cone components: $(v^m \gamma_m)_{\pm\pm} = \pm v_{\pm\pm}$. The anticommutation relations of the flat superspace derivatives are then simply

$$(D_\pm)^2 = \pm i\partial_{\pm\pm}, \qquad \{D_+, D_-\} = 0$$

and the Lorentz generator $M_A{}^B$ acts on spinors and vectors as

$$[M, \chi_\pm] = \pm\tfrac{1}{2}\chi_\pm, \qquad [M, V_{\pm\pm}] = \pm V_{\pm\pm}.$$

For purposes of comparison with other work, e.g., [10], which work in Euclidean space rather than Minkowski space as here, we note that after analytic continuation, $\theta^+ \to \theta$, $\theta^- \to \bar{\theta}$, $i\partial_{++} \to \partial_z$, $-i\partial_{--} \to \partial_{\bar{z}}$.

Appendix B: $D = 2$ Supergravity in Superspace

In this appendix we describe aspects of 2-dimensional supergravity. The covariant derivatives $\nabla_A \equiv E_A + \phi_A M$, $E_A \equiv E_A{}^M D_M$, $D_M = (\partial_m, D_\mu)$ are defined in Appendix A. Their transformation is defined by

$$\delta\nabla_A = [\nabla_A, K] \tag{B.1}$$

with

$$K = K^M D_M + \Lambda M \tag{B.2}$$

which implies

$$\delta E_A{}^N = -K^M D_M E_A{}^N - \Lambda M_A{}^B E_B{}^N + E_A{}^M D_M K^N - 2iE_A{}^\mu K^\nu \bar{\gamma}^n_{\mu\nu} \delta_n{}^N. \tag{B.3}$$

The last term comes from the torsion of flat superspace. In particular we have

$$\begin{aligned}
\delta E_\pm{}^\pm &= -(K^M D_M E_\pm{}^\pm \pm \tfrac{1}{2}\Lambda E_\pm{}^\pm) + E_\pm{}^M D_M K^\pm \\
\delta E_\pm{}^{\pm\pm} &= -(K^M D_M E_\pm{}^{\pm\pm} \pm \tfrac{1}{2}\Lambda E_\pm{}^{\pm\pm}) + E_\pm{}^M D_M K^{\pm\pm} \mp 2iE_\pm{}^\pm K^\pm \\
\delta E_\pm{}^{\mp\mp} &= -(K^M D_M E_\pm{}^{\mp\mp} \pm \tfrac{1}{2}\Lambda E_\pm{}^{\mp\mp}) + E_\pm{}^M D_M K^{\mp\mp} \pm 2iE_\pm{}^\mp K^\mp.
\end{aligned} \tag{B.4}$$

In flat superspace, $E_A{}^M = \delta_A{}^M$, and hence we can choose five nonsingular gauges. Three are algebraic:

(a) Using the Lorentz transformation (Λ)

$$E_+{}^+ = E_-{}^- \tag{B.5}$$

(b) Using spinor translations $(K^\pm)$

$$E_\pm{}^{\pm\pm} = 0. \tag{B.6}$$

Furthermore, there are two gauge choices with spinor derivatives $E_\pm$ of the vector translation parameters $K^{\pm\pm}(E_\pm K^{\mp\mp})$:

$$E_\pm{}^{\mp\mp} = 0. \tag{B.7}$$

We now consider the torsion constraints of $D=2$, $N=1$ supergravity. The torsions and curvatures were defined in Appendix A:

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C\nabla_C + R_{AB}M. \tag{B.8}$$

They satisfy Bianchi identities that follow from the graded Jacobi identities:

$$[[\nabla_{[A}, \nabla_B\}, \nabla_{C)}\} = 0. \tag{B.9}$$

We choose the following constraints:

$$(\nabla_\pm)^2 = \pm i\nabla_{\pm\pm} \Leftrightarrow T_{\pm\pm}{}^A = \pm 2i\delta_{\pm\pm}{}^A, \qquad R_{\pm\pm} = 0 \tag{B.10}$$

$$\{\nabla_+, \nabla_-\} = RM \Leftrightarrow T_{+-}{}^A = 0. \tag{B.11}$$

We show below that these are conventional. The constraint (B.10) includes a constraint on a curvature ($R_{\pm\pm}=0$) as well as on torsions; as is well known [11], superspace constraints can *always* be expressed in terms of torsions only, but the form (B.10)–(B.11) is particularly convenient for expressing all torsions and curvatures in terms of a single irreducible set. This procedure, called "solving the Bianchi identities," is most easily carried out by working directly with the commutator algebra of the covariant derivatives, as the Bianchi identities are highly redundant. Thus we need to determine $[\nabla_\pm, \nabla_{\pm\pm}]$, $[\nabla_\pm, \nabla_{\mp\mp}]$, and $[\nabla_{++}, \nabla_{--}]$. We begin with

$$[\nabla_\pm, \nabla_{\pm\pm}] = \mp i[\nabla_\pm, \nabla_\pm{}^2] = 0 \Leftrightarrow T_{\pm,\pm\pm}{}^A = R_{\pm,\pm\pm} = 0. \tag{B.12}$$

Next, we consider

$$\begin{aligned}
[\nabla_\pm, \nabla_{\mp\mp}] &= \pm i[\nabla_\pm, \nabla_\mp{}^2] = \pm i[\{\nabla_\pm, \nabla_\mp\}, \nabla_\mp] \\
&= \pm i[RM, \nabla_\mp] = \pm i(\mp\tfrac{1}{2}R\nabla_\mp - (\nabla_\mp R)\, M) \\
&= -\frac{i}{2}R\nabla_\mp \mp i(\nabla_\mp R)\, M \\
&\Leftrightarrow T_{\pm,\mp\mp}{}^A = -\frac{i}{2}R\delta_\mp{}^A, \qquad R_{\pm,\mp\mp} = \mp i\nabla_\mp R.
\end{aligned} \tag{B.13}$$

Finally, we have

$$\begin{aligned}
[\nabla_{++}, \nabla_{--}] &= -i[\nabla_+{}^2, \nabla_{--}] = -i\{\nabla_+, [\nabla_+, \nabla_{--}]\} \\
&= -i\left\{\nabla_+, -\frac{i}{2}R\nabla_- - i(\nabla_- R)\, M\right\}
\end{aligned}$$

$$= -i\left[-\frac{i}{2}(\nabla_+ R)\nabla_- - \frac{i}{2}R^2 M - i(\nabla_+\nabla_- R)M - \frac{i}{2}(\nabla_- R)\nabla_+\right]$$
$$= -\tfrac{1}{2}[(\nabla_+ R)\nabla_- + (\nabla_- R)\nabla_+] - (\nabla_+\nabla_- R + \tfrac{1}{2}R^2)M$$
$$\Leftrightarrow T_{++,--}{}^A = -\tfrac{1}{2}(\delta_-{}^A\nabla_+ R + \delta_+{}^A\nabla_- R),$$
$$R_{++,--} = -(\nabla_+\nabla_- R + \tfrac{1}{2}R^2). \tag{B.14}$$

This determines all torsions and curvatures in terms of the single superfield R (note that R is pure imaginary). When this superfield vanishes, the superspace geometry is entirely flat. We see that the constraints (B.10)–(B.11) imply the (redundant set of) purely *torsion* constraints:

$$T_{\pm\pm}{}^A = \pm 2i\delta_{\pm\pm}{}^A, \qquad T_{+-}{}^A = T_{++,--}{}^a = 0 \tag{B.15}$$
$$T_{\pm,\pm\pm}{}^A = T_{\pm,\mp\mp}{}^a = 0. \tag{B.16}$$

In fact, (B.15), which are (2.1)–(2.3) in spinor notation, are sufficient to imply (B.10)–(B.11), and hence (B.12)–(B.14) and (B.16).

We now show that the constraints (B.10)–(B.11) are conventional, that is, given an arbitrary covariant derivative $\tilde{\nabla}_A$ (equivalently, given arbitrary $\tilde{E}_A{}^M$ and $\tilde{\phi}_A$), which define arbitrary unconstrained torsions $\tilde{T}$ and curvatures $\tilde{R}$, we can always find a derivative ∇_A (equivalently, we can find $E_A{}^M$ and ϕ_A expressed in terms of $\tilde{E}_A{}^M$ and $\tilde{\phi}_A$) that satisfy (B.10)–(B.11) and hence (B.12)–(B.14). The full nonlinear computation is messy and unilluminating, so we will only consider $\tilde{\nabla}_A$ that deviate infinitesimally from ∇_A, and torsions and curvatures that deviate infinitesimally from (B.10)–(B.11):

$$(\tilde{\nabla}_\pm)^2 = \pm i\tilde{\nabla}_{\pm\pm} + \tfrac{1}{2}(\delta T_{\pm\pm}{}^A)\tilde{\nabla}_A + \tfrac{1}{2}\delta R_{\pm\pm}M \tag{B.17}$$
$$\{\tilde{\nabla}_+, \tilde{\nabla}_-\} = (\delta T_{+-}{}^A)\tilde{\nabla}_A + \tilde{R}M. \tag{B.18}$$

A straightforward computation shows that

$$\nabla_\pm = \tilde{\nabla}_\pm + (\delta A_{\pm\pm})\tilde{\nabla}_\mp + (\delta C_\pm)M$$
$$\begin{aligned}\nabla_{\pm\pm} = \tilde{\nabla}_{\pm\pm} \mp i[&\tfrac{1}{2}(\delta T_{\pm\pm}{}^A)\nabla_A + \tfrac{1}{2}(\delta R_{\pm\pm})M + (\nabla_\pm\delta A_{\pm\pm})\nabla_\mp \\ &\pm\tfrac{1}{2}(\delta C_\pm)\nabla_\pm + (R\delta A_{\pm\pm} + (\nabla_\pm\delta C_\pm))M]\end{aligned} \tag{B.19}$$

satisfy (B.10)–(B.11) to order δ when

$$\begin{aligned}\delta A_{\pm\pm} &= \mp i\delta T_{+-}{}^{\mp\mp} \\ \delta C_\pm &= \pm 2(\delta T_{+-}{}^{\mp} + \nabla_\mp\delta A_{\pm\pm}).\end{aligned} \tag{B.20}$$

We now solve the constraints (B.10)–(B.11) in terms of unconstrained prepotentials. Previous solutions have been given only in special coordinate systems [4]. Equation (B.10) $\nabla_{\pm\pm} = \mp i(\nabla_\pm)^2$ clearly determines the vector vielbein $E_{\pm\pm}$ and connection $\phi_{\pm\pm}$ in terms of the spinor vielbein $E_{\pm}$ and connection $\phi_{\pm}$. The remaining relation among these quantities follow from (B.11), which splits into two equations:

$$R = E_{+}\phi_{-} + E_{-}\phi_{+} - \phi_{+}\phi_{-} \tag{B.21}$$

$$\{E_{+}, E_{-}\} = \tfrac{1}{2}(\phi_{+}E_{-} - \phi_{-}E_{+}). \tag{B.22}$$

Equation (B.21) determines the superfield R, but (B.22) is a genuine constraint on $E_{\pm}$ and $\phi_{\pm}$. We solve for these quantities in terms of unconstrained prepotentials which we take as the six components of the noncovariant spinor derivatives

$$\check{E}_{\pm} = \check{E}_{\pm}{}^{\pm} D_{\pm} + \check{E}_{\pm}{}^{m}\partial_{m}. \tag{B.23}$$

Note that by definition $\check{E}_{\pm}{}^{\mp} = 0$. From $\check{E}_{\pm}$ we define noncovariant vector derivatives:

$$\check{E}_{\pm\pm} = \mp i(\check{E}_{\pm})^{2} \tag{B.24}$$

and anholonomy coefficients $\check{C}_{AB}{}^{C}$

$$[\check{E}_{A}, \check{E}_{B}\} = \check{C}_{AB}{}^{C}\check{E}_{C}. \tag{B.25}$$

We express $E_{\pm}$ in terms of $\check{E}_{\pm}$; since the prepotentials are unconstrained, the most general $E_{\pm}$ can always be written as

$$E_{\pm} = \check{E}_{\pm} + F_{\pm\pm}\check{E}_{\mp}. \tag{B.26}$$

We substitute (B.26) into (B.22) and find

$$\begin{aligned}(1 + F_{++}F_{--})\{\check{E}_{+}, \check{E}_{-}\} &+ (E_{+}F_{--})\,\check{E}_{+} \\ &+ (E_{-}F_{++})\,\check{E}_{-} + 2iF_{--}\check{E}_{++} - 2iF_{++}\check{E}_{--} \\ &= \tfrac{1}{2}\phi_{+}(\check{E}_{-} + F_{--}\check{E}_{+}) - \tfrac{1}{2}\phi_{-}(\check{E}_{+} + F_{++}\check{E}_{-}).\end{aligned} \tag{B.27}$$

From the coefficients of $\check{E}_{\pm\pm}$ we find

$$\pm 2iF_{\mp\mp} + (1 + F_{++}F_{--})\,\check{C}_{+-}{}^{\pm\pm} = 0 \tag{B.28}$$

which implies

$$F_{\pm\pm} = \mp\frac{i}{\check{C}_{+-}{}^{\pm\pm}}\left(1 - \sqrt{1 - \check{C}_{+-}{}^{++}\check{C}_{+-}{}^{--}}\right) = \mp\frac{i}{2}\check{C}_{+-}{}^{\mp\mp} + \mathcal{O}(C^{3}) \tag{B.29}$$

and determines $E_{\pm}$ in terms of the prepotentials. For the coefficients of $\check{E}_{\pm}$, after some algebra, we find

$$\phi_{\pm} = \pm 2\,\frac{1 + F_{++}F_{--}}{1 - F_{++}F_{--}}\,[\check{C}_{+-}{}^{\mp} + E_{\mp}F_{\pm\pm} - F_{\pm\pm}(\check{C}_{+-}{}^{\pm} + E_{\pm}F_{\mp\mp})]. \tag{B.30}$$

We have thus found $\nabla_{\pm}$ in terms of the prepotentials, and, from $\nabla_{\pm\pm} = \mp i(\nabla_{\pm})^2$, all of ∇_A.

In the conformal gauge, $E_{\pm} = \check{E}_{\pm}$ and $\phi_{\pm} = \pm 2C_{+-}{}^{\mp}$; in particular we see explicitly that the gauge condition $\check{E}_{\pm} = e^{\psi} D_{\pm}$ implies $F_{\pm\pm} = 0$ and hence $E_{\pm}{}^{\mp} = 0$.

The formalism developed here can be used to simplify some of the calculations in Sections 2–4. In particular, the gauge conditions (B.5)–(B.7), imposed on the "twiddled" variables (cf. (3.18)), are

$$E^{-3/2} E_{\pm}{}^{\mp\mp} = 0, \qquad E^{-3/2} E_{\pm}{}^{\pm\pm} = 0, \qquad E^{-1}(E_{+}{}^{+} - E_{-}{}^{-}) = 0. \tag{B.31}$$

This form avoids the problem of separating out pieces of $E_{\alpha}{}^{m}$ as discussed after (2.15). From the variations (B.4), using the rescaled variables (3.17) and the conformal gauge (B.31) (with $E = e^{2\psi}$), we find the ghost Lagrangian (cf. (3.19)).

$$\begin{aligned} \mathscr{L}_{\text{ghost}} = {} & \tilde{\bar{C}}_{\mp\mp}{}^{\pm} e^{-2\psi} D_{\pm}(e^{3\psi} \tilde{C}^{\mp\mp}) \\ & + \tilde{\bar{C}}_{\pm\pm}{}^{\pm} e^{-2\psi} [D_{\pm}(e^{3\psi} \tilde{C}^{\pm\pm}) \mp 2ie^{2\psi} \tilde{C}^{\pm}] \\ & + \tilde{\bar{C}} e^{-\psi} [-e^{\psi} \tilde{C} + D_{+}(e^{2\psi} \tilde{C}^{+}) - D_{-}(e^{2\psi} \tilde{C}^{-})] \end{aligned} \tag{B.32}$$

with summation over all $\pm$. The matter lagrangian remains $L_{\text{matter}} = 2D_{+}(e^{\psi} \tilde{X}^{i}) D_{-}(e^{\psi} \tilde{X}^{i})$ as in (3.19). We perform the same rescalings as in (4.1). The discussion for the matter fields proceeds unmodified ((4.2)–(4.8)). However, the ghost sector simplifies. For the same reasons as discussed below (4.8), we only need to keep one term from (B.32) (cf. 4.9):

$$\tilde{\bar{C}}_{--}{}^{+} e^{-2\psi} D_{+} e^{3\psi} \tilde{C}^{--} + \tilde{\bar{C}}_{++}{}^{-} e^{-2\psi} D_{-} e^{3\psi} \tilde{C}^{++}. \tag{B.33}$$

The kinetic operator in (B.33) is not hermitian. As explained below (4.9), we overcome this problem by replacing a lagrangian $\bar{C}\mathbf{O}C$ with $C\mathbf{O}^{\dagger}\mathbf{O}C + \bar{C}\mathbf{O}\mathbf{O}^{\dagger}\bar{C}$. Then (B.33) is replaced by

$$\begin{aligned} & \tilde{\bar{C}}_{--}{}^{+}(e^{-2\psi} D_{+} e^{3\psi})(e^{3\psi} D_{-} e^{-2\psi}) \tilde{\bar{C}}_{++}{}^{-} + \tilde{C}^{++}(e^{3\psi} D_{-} e^{-2\psi})(e^{-2\psi} D_{+} e^{3\psi}) \tilde{C}^{--} \\ & \quad = \tilde{\bar{C}}_{--}{}^{+} e^{-2\psi} D_{+} e^{6\psi} D_{-} e^{-2\psi} \tilde{\bar{C}}_{++}{}^{-} + \tilde{C}^{++} e^{3\psi} D_{-} e^{-4\psi} D_{+}{}^{3\psi} \tilde{C}^{--}. \end{aligned} \tag{B.34}$$

The operators in (B.34)

$$\begin{pmatrix} 0 & e^{-2\psi} D_{+} e^{6\psi} D_{-} e^{-2\psi} \\ e^{-2\psi} D_{-} e^{6\psi} D_{+} e^{-2\psi} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & e^{3\psi} D_{-} e^{-4\psi} D_{+} e^{3\psi} \\ e^{3\psi} D_{+} e^{-4\psi} D_{-} e^{3\psi} & 0 \end{pmatrix} \tag{B.34}$$

are hermitian and can be used to regularize the trace just as we did for the scalar case, without any of the complications described in (4.13)–(4.14), and we obtain (4.19)–(4.21) directly.

To compute, for example, the superstress tensor (supercurrent) of various systems, it is useful to give the complete solution of the vielbein to linear order. Linearizing (B.29)–(B.30), we find

$$\nabla_\pm = \check{E}_\pm \mp \frac{i}{2}(D_+ \check{E}_-{}^{\mp\mp} + D_- \check{E}_+{}^{\mp\mp}) D_\mp$$
$$\pm (iD_+ D_- \check{E}_\mp{}^{\mp\mp} - \partial_{\mp\mp} \check{E}_\pm{}^{\mp\mp} + 2D_\pm \check{E}_\mp{}^{\mp}) M + \mathcal{O}(\delta E^2)$$

and

$$\operatorname{sdet} E_A{}^M = 1 + \operatorname{str}(\delta E_A{}^M) + \theta(\delta E^2) \tag{B.35}$$

$$= 1 + \frac{i}{2}(D_- \check{E}_-{}^{--} - D_+ \check{E}_+{}^{++}) - \check{E}_+{}^+ - \check{E}_-{}^- + \theta(\delta E^2). \tag{B.36}$$

where δE means all linear terms.

It is also interesting to find how the x-space components lie within the vielbein. We use a *nontrivial* modification of standard techniques [9] to find a Wess–Zumino gauge choice in which the conformal gauge choices in superspace ($E_\alpha = e^\psi D_\alpha$) and in x space ($e_a{}^m = \rho \delta_a{}^m$, $\psi_a = \gamma_a \chi$) are compatible. In the usual procedure, one fixes a Wess–Zumino gauge $\nabla_\alpha| = \partial_\alpha$, which is clearly incompatible with the superspace conformal gauge choice. We therefore begin with a gauge choice

$$\begin{aligned} E_\alpha| &= e^{-1/4}\partial_\alpha, \qquad & E_a| &= e_a + e^{-1/4}\psi_a{}^\mu \partial_\mu \\ \partial_\alpha &\equiv \delta_\alpha{}^\mu \partial_\mu, \qquad & e_a &= e_a{}^m \partial_m, \qquad e = \det e_m{}^a \end{aligned} \tag{B.37}$$

where $X|$ denotes the θ-independent projection of X. The powers of e in (B.37) follow directly from the condition that the component conformal gauge $e_a{}^m = e^{-1/2}\delta_a{}^m$, $\psi_a{}^\mu = (\gamma_a \chi) \leftrightarrow \psi_{\pm\pm}{}^\mp = 0$, be compatible with the superspace conformal gauge (B.5)–(B.7):

$$E_\pm = e^\psi D_\pm \Rightarrow E_{\pm\pm} = e^{2\psi}\partial_{\pm\pm} \mp ie^{2\psi}(D_\pm \psi) D_\pm. \tag{B.38}$$

Then taking (B.37), substituting into the constraints (B.22), and imposing the compatibility of (B.38) with the component conformal gauge, a straightforward computation gives

$$\begin{aligned} \nabla_\pm = {} & e^{-1/4}\partial_\pm + i\psi_{\pm\pm}{}^\pm M \pm i\theta^\pm (e^{1/4} D_{\pm\pm} + \tfrac{1}{2}\psi_{\pm\pm}{}^\pm \partial_\pm + \psi_{\pm\pm}{}^\mp \partial_\mp) \\ & \mp i\theta^\mp \psi_{\mp\mp}{}^\mp \partial_\pm + \tfrac{1}{2}\theta^\mp e^{1/4}(iS + \psi_{--}{}^- \psi_{++}{}^+) M \\ & + \theta^+\theta^- \Big[\tfrac{1}{4} e^{1/4}(iS + \psi_{--}{}^- \psi_{++}{}^+ + 4\psi_{++}{}^- \psi_{--}{}^+) \partial_\pm \\ & \qquad - \frac{i}{2} e^{1/4}(\tfrac{1}{2} e_{\pm\pm} \ln e \pm \phi_{\pm\pm}) \partial_\mp \\ & \qquad \pm \tfrac{1}{2} e^{1/2} \psi_{\mp\mp}{}^\mp e_{\pm\pm} \pm e^{1/2} \psi_{\pm\pm}{}^\mp e_{\mp\mp} + f_\pm M \Big] \end{aligned} \tag{B.39}$$

where the component connection $\phi_{\pm\pm}$ has torsion:

$$\phi_{\pm\pm}=(e_{++}e_{--}{}^{m}-e_{--}e_{++}{}^{m})\,e_{m}{}^{\mp\mp}+i\psi_{\mp\mp}{}^{\mp}\psi_{\pm\pm}{}^{\mp}. \tag{B.40}$$

S is the x-space auxiliary field of the supergravity multiplet and $f_{\pm}$ is easily determined but uninteresting. Note that in component conformal gauge, $\phi_{\pm\pm}=-\frac{1}{2}e_{\pm\pm}\ln e$ and $E_{\pm}$ reduces to

$$E_{\pm}=\left[e^{-1/4}+\frac{i}{2}(\theta^{+}\psi_{++}{}^{+}-\theta^{-}\psi_{--}{}^{-})+\frac{1}{4}\theta^{+}\theta^{-}e^{1/4}(iS+\psi_{--}{}^{-}\psi_{++}{}^{+})\right]D_{\pm}. \tag{B.41}$$

One can also compute the vector derivative and superdeterminant, and find the superspace parameters of component supersymmetry transformations, but we will find these from the gauge completion procedure described in Appendix C.

Appendix C: Gauge Completion

In Section 2, we have chosen a particular gauge, namely the conformal gauge (2.5) for the super vielbein $E_{A}{}^{M}$. In this appendix, we discuss the problem of gauge completion, i.e., we identify the supervielbein in (2.5) with the corresponding fields in the x-space supergravity multiplet, namely the vielbein $e_{a}{}^{m}$, the gravitino $\psi_{a}{}^{\mu}$, and the auxiliary field S.

The usual gauge completion program proceeds as follows [6]: One starts by identifying the θ^{0} component of the superparameter K^{M}, Λ^{AB}, and supervielbein $E_{A}{}^{M}$ with the x-space parameter and fields:

$$\begin{aligned} &K^{m}|_{\theta=0}=\xi^{m}, && E_{a}{}^{m}|_{\theta=0}=e_{a}{}^{m},\\ &K^{\mu}|_{\theta=0}=\varepsilon^{\alpha}\delta_{\alpha}{}^{\mu}, && E_{a}{}^{\mu}|_{\theta=0}=-\psi_{a}{}^{\mu},\\ &\Lambda^{ab}|_{\theta=0}=\lambda^{ab}. \end{aligned} \tag{C.1}$$

At higher θ components, one requires the superspace parameter composition law to be compatible with that of x space, and the superspace transformation law of the supervielbein to be compatible with the x-space transformation law. Working this way, one finds all the θ components of the superparameter and the supervielbein in terms of x-space fields.

The gauge completion program with (C.1) is fairly easy to carry out [6]. But, the result is not compatible with our superspace gauge choice (2.5). For example, one always gets $E_{\alpha}{}^{\mu}(x,\theta=0)=\delta_{\alpha}{}^{\mu}$ by gauge completion, but the gauge choice in (2.5) is $E_{\alpha}{}^{\mu}=e^{\psi}\delta_{\alpha}{}^{\mu}$. Therefore, we have to modify (C.1) in such a way as to make the gauge completion compatible with the (2.5). To do this, we first make a general ansatz for

K^m, K^μ, and Λ^{ab} and fix their form by requiring the superspace parameter composition law

$$K_{12}{}^M = K_2{}^N D_N K_1{}^M - K_1{}^N D_N K_2{}^M + \bar{\delta}_1 K_2{}^M - \bar{\delta}_2 K_1{}^M + a K_2{}^\mu K_1{}^\nu \bar{\gamma}^m_{\mu\nu} \delta_m{}^M \quad \text{(C.2)}$$

$$\begin{aligned} \Lambda_{12}{}^{ab} = {} & K_2{}^N D_N \Lambda_1{}^{ab} - K_1{}^N D_N \Lambda_2{}^{ab} + \Lambda_2{}^{ac} \Lambda_{1c}{}^b \\ & - \Lambda_1{}^{ac} \Lambda_{2c}{}^b + \bar{\delta}_1 \Lambda_2{}^{ab} - \bar{\delta}_2 \Lambda_1{}^{ab} \end{aligned} \quad \text{(C.3)}$$

to be compatible with the x-space parameter composition law

$$\xi_{12}{}^m = \xi_2{}^n \partial_n \xi_1{}^m + \tfrac{1}{4} \bar{\varepsilon}_2 \gamma^m \varepsilon_1 - (1 \leftrightarrow 2) \quad \text{(C.4)}$$

$$\varepsilon_{12}{}^\alpha = \xi_2{}^n \partial_n \varepsilon_1{}^\alpha + \tfrac{1}{4} \lambda_2{}^{ab} (\gamma_{ab} \varepsilon_1)^\alpha - \tfrac{1}{4} (\bar{\varepsilon}_2 \gamma^m \varepsilon_1) \psi_m{}^\alpha - (1 \leftrightarrow 2) \quad \text{(C.5)}$$

$$\lambda_{12}{}^{ab} = \xi_2{}^n \partial_n \lambda_1{}^{ab} + \lambda_2{}^{ac} \lambda_{1c}{}^b + \tfrac{1}{4} (\bar{\varepsilon}_2 \gamma^m \varepsilon_1) \omega_m{}^{ab} + \tfrac{1}{8} S (\bar{\varepsilon}_2 \gamma^{ab} \varepsilon_1) - (1 \leftrightarrow 2) \quad \text{(C.6)}$$

where $\bar{\delta}_1 K_2{}^M$ means the variation of the field in the parameter $K_2{}^M$ and the last term in (C.2) is due to the anticommutator of D_μ and D_ν[1]:

$$\{D_\mu, D_\nu\} = -a \bar{\gamma}^m_{\mu\nu} \partial_m. \quad \text{(C.7)}$$

The constant a is also fixed by the compatibility requirement. At $\theta = 0$, we make the following identification:

$$\begin{aligned} & K^m(\theta = 0) = \xi^m, \qquad K^\mu(\theta = 0) = e^p \varepsilon^\alpha \delta_\alpha{}^\mu \\ & \Lambda^{ab}(\theta = 0) = \lambda^{ab} + \tfrac{1}{4} (\bar{\varepsilon} \gamma_5 \gamma \cdot \psi) \varepsilon^{ab} \end{aligned} \quad \text{(C.8)}$$

where $e = \det e_m{}^a$. For $K^m|$, the most general identification is $K^m| = e^q \xi^m$, but it turns out that compatibility in the parameter composition law requires $q = 0$. For $K^\mu|$ however, it is consistent to keep $p \neq 0$ and it turns out that $p = -\frac{1}{4}$ in conformal gauge. The last term in Λ^{ab} $(\theta = 0)$ is very important; it plays the role of a "compensating" Lorentz transformation to keep one in the conformal gauge. (The factor $\frac{1}{4}$ is fixed by restricting the vielbein to be of the form (C.9), see below.)

We now look at the spinor vielbein $E_\alpha{}^M$. Since we know that $E_\alpha{}^\mu = e^\psi$ and $E_\alpha{}^m = 0$ in conformal gauge we make the most general ansatz for $E_\alpha{}^m$ at the θ^1 level by adding terms that vanish in conformal gauge:

$$\begin{aligned} & E_\alpha{}^m = 0 + h(\bar{\theta} \gamma^m)_\alpha - h e^{-1/2} (\bar{\theta} \bar{\gamma}^m)_\alpha \\ & E_\alpha{}^\mu \delta_\mu{}^\beta = e^p \delta_\alpha{}^\beta + v(\bar{\theta} \gamma \cdot \psi) \delta_\alpha{}^\beta + w (\gamma^n)_\alpha{}^\beta (\bar{\psi}_m \gamma_n \gamma^m \theta). \end{aligned} \quad \text{(C.9)}$$

[1] In this appendix we use the conventions of [6], where the x-space formulation is given. These conventions differ from those used elsewhere in this paper, which may be recovered by Wick-rotating and rescaling the coordinates and derivatives appropriately.

This choice strongly restricts the θ^1 terms in K^m, and K^μ. We find

$$\begin{aligned} K^m &= \xi^m + \tfrac{1}{4}e^{1/4}(\bar\theta\gamma^m\varepsilon) + \tfrac{1}{4}e^{-1/4}(\bar\theta\bar\gamma^m\varepsilon) \\ K^\mu\delta_\mu{}^\alpha &= e^{-1/4}\varepsilon^\alpha + \tfrac{1}{4}(\partial_m\xi^m)\,\theta^\alpha - \tfrac{1}{8}(\bar\varepsilon\gamma^m\gamma^n\psi_m)(\gamma_n\theta)^\alpha \\ &\quad + \tfrac{1}{8}(\bar\varepsilon\gamma_5\gamma\cdot\psi)(\gamma_5\theta)^\alpha - \tfrac{1}{4}\lambda^{ab}(\gamma_{ab}\theta)^\alpha \end{aligned} \tag{C.10}$$

and

$$a = \tfrac{1}{2}, \qquad h = -\tfrac{1}{4}e^{1/4}, \qquad v = w = -\tfrac{1}{8}.$$

Having determined the θ^1 component of the parameters, it is then straightforward to evaluate any components of the vielbein to θ^1 order. At θ^2 order, it is tedious and unilluminating to calculate everything explicitly. However, sdet E is a very useful object, since the supervielbein appears in the measure only through sdet E. Therefore we determine the full θ expansion of E. Under supercoordinate transformations, E transforms as:

$$\delta E = K^N D_N E - (-)^N (D_N K^N)\,E. \tag{C.11}$$

Using (C.10) we can easily determine E to the θ^1 level:

$$E = e^{-1/2} - \tfrac{1}{4}e^{-1/4}(\bar\theta\gamma\cdot\psi). \tag{C.12}$$

As one sees from (C.11) one only has to work out the θ^2 piece in K^μ to determine E to order θ^2. We find

$$\begin{aligned} K^\mu(\theta^2 \text{ level})\,\delta_\mu{}^\alpha = e^{1/4}[&-\tfrac{1}{16}(e_a{}^m\partial_n e_m{}^a)(\bar\theta\gamma^n\varepsilon)\,\theta^\alpha \\ &+ \tfrac{1}{16}(\bar\theta\gamma^m\varepsilon)(\bar\theta\gamma^n\psi_m)\,\psi_n{}^\alpha \\ &- \tfrac{1}{16}\theta^2\omega_m{}^{ab}(e,\psi)\,e_a{}^m(\gamma_b\varepsilon)^\alpha] \end{aligned} \tag{C.13}$$

and

$$\begin{aligned} E = e^{-1/2} &- \tfrac{1}{4}e^{-1/4}(\bar\theta\gamma\cdot\psi) + \tfrac{1}{16}S\theta^2 \\ &+ \tfrac{1}{32}(\bar\theta\gamma\cdot\psi)(\bar\theta\gamma\cdot\psi) + \tfrac{1}{64}(\bar\psi_m\gamma^n\gamma^m\psi_n)\,\theta^2. \end{aligned} \tag{C.14}$$

APPENDIX D: A LEMMA FOR REGULATED SUPERTRACES

In this appendix we prove the following lemma:

$$\begin{aligned} L &= \lim_{M\to\infty}\int \frac{d^2k}{(2\pi)^2}\,d^2\chi\,e^{-iZ\cdot K}e^{H/M^2}e^{iZ\cdot K} \\ &= \frac{i}{2\pi}(\text{sdet}\,g^{MN})^{-1/2} = \frac{i}{8\pi}(\det g^{mn})^{-1/2}(g^{\mu\nu} - g_{mn}g^{m\mu}g^{n\nu})\,\varepsilon_{\mu\nu} \end{aligned} \tag{D.1}$$

where $H = g^{MN}\partial_N\partial_M + V^M\partial_M + X$ with g, V, and X arbitrary superfields, $Z\cdot K \equiv$

$Z^M K_M = x^m k_m - i\theta^\mu \chi_\mu$ is real. We first pull the superplane wave $e^{iZ \cdot K}$ through the operator H:

$$e^{-iZ \cdot K} H e^{iZ \cdot K} = H + iV^M K_M + 2ig^{MN} K_M \partial_N - g^{MN} K_M K_N. \tag{D.2}$$

We now rescale $K_M \to MK_M$; because of supersymmetry, and in contrast to the bosonic case [1, 2], the measure $d^2k d^2\chi$ *is preserved* by this rescaling, and we get

$$L = \lim_{M \to \infty} \int \frac{d^2k}{(2\pi)^2} d^2\chi \exp\left[\left(\frac{H}{M^2} + \frac{i}{M}(V^M K_M + 2g^{MN} K_N \partial_M) - g^{MN} K_N K_M\right]. \tag{D.3}$$

Because no factors of M came out after the rescaling of k, there are no subtleties involved in taking the $M \to \infty$ limit, and we can drop the first two terms in (D.3) to obtain

$$L = \int \frac{d^2k}{(2\pi)^2} d^2\chi\, e^{-g^{MN} K_N K_M} = \frac{i}{2\pi} (\operatorname{sdet} g^{MN})^{-1/2}. \tag{D.4}$$

The result (D.1) follows immediately (the simplified form of the superdeterminant in the last expression in (D.1) follows from the identity $g^{\mu\nu} - g_{mn} g^{m\mu} g^{n\nu} = \frac{1}{2}\varepsilon^{\mu\nu}\varepsilon_{\rho\sigma}(g^{\rho\sigma} - g_{mn} g^{m\rho} g^{n\sigma})$). A final comment: If we had started with our kinetic operator H in covariant form, $H = \hat{g}^{MN} D_N D_M + \cdots$, though $\hat{g}^{MN} \neq g^{MN}$, $\operatorname{sdet} \hat{g}^{MN} = \operatorname{sdet} g^{MN}$ and we would get the same answer. This follows because the shift $\partial_M| \to D_M|$ can be compensated by a shift $\chi_\mu \to \chi_\mu + (\theta k)_\mu$ in (D.4), which is a unimodular transformation.

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Progress towards the supersymmetrization of Chern-Simons terms in five-dimensional simple supergravity

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The supersymmetric generalization of the mixed Chern-Simons term $F\wedge \mathrm{tr}(\omega\wedge R-\frac{1}{3}\omega\wedge\omega\wedge\omega)$ in five dimensions is constructed to the lowest two orders in the gravitational coupling constant. It is shown that the supersymmetrization is only possible when one modifies the fermionic transformation rules by adding purely bosonic terms.

I. INTRODUCTION

Chern-Simons (CS) terms play an important role in supergravity (for a review see Ref. 1). In particular, in $d=10$ dimensions, the coupling of $N=1$ Maxwell matter to $N=1$ supergravity[2] can be extended to a Yang-Mills coupled system[3] by extending the combination $B_{[\mu}F_{\nu\rho]}$ in the action to a Yang-Mills CS term

$$\omega^{\mathrm{YM}}_{\mu\nu\rho}\equiv \mathrm{tr}(B_{[\mu}F_{\nu\rho]}-\tfrac{2}{3}B_{[\mu}B_{\nu}B_{\rho]}) , \tag{1.1}$$

where B_μ is the Lie-algebra-valued connection and $F_{\nu\rho}$ its curl. By adding to this Yang-Mills CS term the corresponding gravitational CS term

$$\omega^{G}_{\mu\nu\rho}\equiv \mathrm{tr}(\omega_{[\mu}R_{\nu\rho]}-\tfrac{2}{3}\omega_{[\mu}\omega_{\nu}\omega_{\rho]}) \tag{1.2}$$

one finds that the one-loop anomalies cancel,[4] provided the Yang-Mills gauge group is either SO(32) or $E_8\times E_8$. The action contains (1.1) and (1.2) in the combination ${H_{\mu\nu\rho}}^2$ where

$$H_{\mu\nu\rho}=(\partial_{[\mu}A_{\nu\rho]}+\tfrac{1}{30}\omega^{\mathrm{YM}}_{\mu\nu\rho}+\omega^{G}_{\mu\nu\rho}) . \tag{1.3}$$

The presence of (1.2) in $H_{\mu\nu\rho}$ violates local supersymmetry. It is unknown at present whether further additions to the action and transformation rules can restore local supersymmetry. It is known, however, from the zero-slope limit of the $d=10$ superstring theory that ${R_{\mu\nu\rho\sigma}}^2$ terms are present.[5] Moreover, one can avoid modifications in the propagators if the R^2 terms in the action appear in the combination

$${R_{\mu\nu\rho\sigma}}^2-4{R_{\mu\nu}}^2+R^2 . \tag{1.4}$$

In $d=4$ dimensions, this is a total derivative, but in $d\neq 4$ dimensions, the terms quadratic in the gravitational fluctuations are still a total derivative.[6] It is known that R^2 terms in the action do not necessarily lead to nonunitarity,[7] but (1.4) has the attraction that it can be written in form language,[8] namely, as

$$R^{ab}\wedge R^{cd}\wedge e^{e_1}\wedge\cdots\wedge e^{e_{n-4}}\epsilon_{abcde_1\cdots e_{n-4}} . \tag{1.5}$$

Recently, it was shown that the $d=3$ supersymmetric gravitational CS term[9] can be written as[10]

$$\gamma_{AB}R^{B}\wedge\omega^{A}+\tfrac{1}{6}f_{ABC}\omega^{C}\wedge\omega^{B}\wedge\omega^{A} , \tag{1.6}$$

where γ_{AB} and f_{ABC} are the Killing supermetric and the structure constants of the superconformal algebra in $d=3$, which is osp(1/4). In fact, (1.6) is just the action of $d=3$ simple conformal supergravity.[10] This result has been extended to the case $N>1$ (Ref. 11). The results in (1.5) and (1.6) point to a general scheme in terms of forms. We note that (1.6) can be written as a supertrace

$$\mathrm{str}(\omega\wedge R-\tfrac{1}{3}\omega\wedge\omega\wedge\omega) , \tag{1.7}$$

where R and ω are superalgebra-valued curvatures and connections. It is tempting to conjecture that in addition to (1.5) there will be terms in $d=10$ with more than two curvatures, but still of the generic form (1.6).

In a recent article,[12] compactification of $d=10$, $N=1$ supergravity was studied with $H_{\mu\nu\rho}$ in (1.3) in the gravitino transformation law. It was thus assumed that there were no extra purely bosonic terms in $\delta\psi_\mu$, or if there were such terms, that they vanished in the Calabi-Yau background studied in Ref. 12. In general, adding bosonic terms to the action will require further fermionic terms to be added for supersymmetry, thus modifying the gravitino field equations. This in turn implies that the gravitino transformation rule must be modified since the supersymmetry algebra closes only modulo fermionic field equations. The question is whether the modifications of $\delta\psi_\mu$ are only fermionic, or perhaps contain also purely bosonic terms. It is this problem which we will study in this paper.

The $d=10$ model is quite complicated, containing a scalar and spin-$\frac{1}{2}$ field, in addition to gauge fields. We therefore look for a simpler system to study the same question, and turn to simple ($N=2$) supergravity in $d=5$ dimensions.[13–15] This model contains only gauge fields: a vielbein ${e_\mu}^m$, a photon B_μ, and a symplectic Majorana gravitino

$$\begin{aligned}&{\bar\psi_\mu}{}^{i}=\psi^{T}_{\mu j}\Omega^{ji}C_5 ,\\ &\bar\epsilon^{i}\gamma_{m_1}\cdots\gamma_{m_k}\psi_{\mu j}=\bar\psi_{\mu j}\gamma_{m_k}\cdots\gamma_{m_1}\epsilon^{i} ,\\ &\bar\epsilon^{i}\psi_{\mu i}=-\bar\epsilon_i{\psi_\mu}^{i} ,\end{aligned} \tag{1.8}$$

where C_5 is the $d=5$ charge-conjugation matrix. In the action one finds a term $F\wedge F\wedge B$. We begin by adding to the term $F\wedge B$ the gravitational CS term of (1.2). We then get a combination as (1.3), namely,

$$F\wedge[F\wedge B+\mathrm{tr}(\omega\wedge R-\tfrac{1}{3}\omega\wedge\omega\wedge\omega)]\,. \tag{1.9}$$

The R- and ω-dependent terms break the local supersymmetry of the action, and we will use the order-by-order-in-κ Noether method to find the corrections in the action and transformation rules needed to restore local supersymmetry at the first two levels in an expansion in the gravitational coupling constant κ.

We begin by noting that after partial integration (1.9) can be written as

$$F\wedge F\wedge B+\mathrm{tr}R\wedge R\wedge B\,. \tag{1.10}$$

We add all possible 5-forms with the same scale.[16] There are only two such terms, namely,

$$\bar\psi\wedge\gamma^{mn}D\psi\wedge R_{mn},\quad R^{mn}\wedge R^{pq}\wedge e^{s}\epsilon_{mnpqs}\,. \tag{1.11}$$

It is interesting to see that the combination (1.5) again arises. One might try to rewrite the sum of the terms in (1.10) and (1.11) as $\mathrm{str}R\wedge R\wedge\omega$, which would predict the coefficients of these terms in (1.10) and (1.11) to be the d_{ABC} symbols, i.e., the supersymmetric generalization of the anomaly coefficients $\mathrm{str}\{\lambda_A,\lambda_B\}\lambda_C$ for the $d=5$ simple anti—de Sitter superalgebra su(2,2 | 1). We will not pursue this idea further in this paper, but instead turn to the simple and direct Noether method.

In Sec. II we shall start from the action

$$\mathscr{L}_{\mathrm{CS}}=a\,\mathrm{tr}R\wedge R\wedge B+b\bar\psi\wedge\gamma^{mn}D\psi\wedge R_{mn}+cR^{mn}\wedge R^{pq}\wedge e^{s}\epsilon_{mnpqs} \tag{1.12}$$

and fix the constants a,b,c such that this action is invariant up to lowest order in κ. Then we determine all variations of $\mathscr{L}_{\mathrm{CS}}$ to the next order in κ. First we show that on shell we can cancel these variations by adding $D\bar\psi D\psi F$ and RFF terms to $\mathscr{L}_{\mathrm{CS}}$. Of course this means that we can cancel $\delta\mathscr{L}_{\mathrm{CS}}$ also off shell by adding extra terms to the transformation rules.

The question we will answer is whether one can cancel $\delta\mathscr{L}_{\mathrm{CS}}$ off shell without extra purely bosonic terms in the $\delta\psi_\mu$ law. In Sec. III we show that it is not possible to cancel $\delta\mathscr{L}_{\mathrm{CS}}$ off shell by only adding terms to the action without modifying the transformation rules. In fact, we show that there are purely bosonic terms $\delta\psi\sim RF\epsilon$ in the gravitino transformation law.

In Sec. IV we show that the $\delta\mathscr{L}_{\mathrm{CS}}\sim D\bar\psi D\psi D\bar\psi\epsilon$ terms vanish on shell. This implies that there are extra fermionic terms in the transformation law of the gravitino, of the form $\delta\psi_\mu\sim D\bar\psi D\psi\epsilon$. This vanishing is interesting because it indicates the existence of a supersymmetric CS term at the third level of κ expansion, but of course, there are further variations at this order that we have not analyzed.

II. ON-SHELL INVARIANCE

By on-shell invariance we mean invariance under the on-shell transformation rules. The $N=2$ supergravity action in five dimensions is given by [throughout this paper, all antisymmetrization is always with strength one, e.g., $\partial_{[\mu}B_{\nu]}\equiv\frac{1}{2}(\partial_\mu B_\nu-\partial_\nu B_\mu)$]

$$\mathscr{L}(e,\psi,B)=-\frac{e}{2\kappa^2}R(\omega_0)-\frac{e}{2}\bar\psi^{i}_{\mu}\gamma^{\mu\rho\sigma}D_\rho\left[\frac{\omega_0+\hat\omega}{2}\right]\psi_{\sigma i}-\tfrac{3}{2}F_{\mu\nu}{}^2-i\kappa\epsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}F_{\rho\sigma}B_\tau$$
$$-\frac{3ie}{16}\kappa(F_{\mu\nu}+\hat F_{\mu\nu})\bar\psi^{i}_{\rho}\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi_{\sigma i}\,, \tag{2.1}$$

where ω_0 is the spin connection obtained by solving the field equation $\delta\mathscr{L}/\delta\omega_0=0$, and $\hat\omega(e,\psi)$ is the supercovariant spin connection as usual,

$$\hat\omega_{\mu mn}(e,\psi)=\omega_{\mu mn}(e)+\frac{\kappa^2}{2}\bar\psi^{i}_{\mu}\gamma_{[m}\psi_{n]i}+\frac{\kappa^2}{4}\bar\psi^{i}_{m}\gamma_\mu\psi_{ni}\,. \tag{2.2}$$

For later use let us introduce yet another spin connection

$$\omega_{\mu mn}(e,\psi,F)=\hat\omega_{\mu mn}(e,\psi)-\frac{i}{2}\kappa\epsilon_{\mu mnab}\hat F^{ab}\,, \tag{2.3}$$

where the supercovariant curvature $\hat F_{ab}$ is defined by

$$\hat F_{\mu\nu}=2\partial_{[\mu}B_{\nu]}+\frac{i}{4}\kappa\bar\psi^{i}_{\mu}\psi_{\nu i}\,. \tag{2.4}$$

The action (2.1) is invariant under the following set of local supersymmetric transformation laws:

$$\delta e^{m}{}_{\mu}=\frac{\kappa}{2}\bar\epsilon^{i}\gamma^{m}\psi_{\mu i},\quad \delta B_\mu=-\frac{i}{4}\bar\epsilon^{i}\psi_{\mu i}\,,$$
$$\delta\psi^{i}_{\mu}=\kappa^{-1}D_\mu(\hat\omega)\epsilon^{i}+\frac{i}{4}\hat F_{\rho\sigma}(\gamma^{\rho\sigma}\gamma_\mu+2\gamma^{\rho}\delta^{\sigma}_{\mu})\epsilon^{i}=\kappa^{-1}D_\mu(\omega)\epsilon^{i}+i\gamma^{\rho}\hat F_{\rho\mu}\epsilon^{i}\,. \tag{2.5}$$

In the $\delta\psi_\mu{}^{i}$ law, we have absorbed part of the F-dependent term into the definition of the connection $\omega(e,\psi,F)$ given by (2.3).

From (2.5) we also derive the transformation laws:

$$\delta\omega_{\mu mn}=-\kappa\bar{\epsilon}^i\gamma_{[n}\rho_{m]\mu i}-\frac{\kappa}{2}\bar{\epsilon}^i\gamma_\mu\rho_{mni}-\frac{\kappa}{4}\epsilon_{\mu mnab}\bar{\epsilon}^i\rho_{abi}+O(\kappa^2)\ ,$$
$$\delta\hat{F}_{\mu\nu}=-\frac{i}{2}\bar{\epsilon}^i\rho_{\mu\nu i}+O(k^1),\ \ \delta\rho_{\mu\nu i}=\frac{1}{8\kappa}R_{\mu\nu ab}\gamma^{ab}\epsilon_i+O(\kappa^0)\ , \tag{2.6}$$

where $\rho_{\mu\nu i}\equiv D_{[\mu}(\omega)\psi_{\nu]i}$ is the gravitino curl. This set of transformation laws is not complete, but it suffices to the order we are working.

Now let us turn to the problem of supersymmetrizing $F\wedge\mathrm{tr}(\omega\wedge R-\frac{1}{3}\omega\wedge\omega\wedge\omega)$. In Sec. I we introduced an ansatz based on the group manifold approach[16]

$$\mathscr{L}_{\mathrm{CS}}=a\kappa^{1/3}R_{\mu\nu}{}^{ab}R_{\rho\sigma ab}B_\tau\epsilon^{\mu\nu\rho\sigma\tau}+b\kappa^{4/3}\bar{\psi}_\mu{}^i\gamma_{mn}R_{\nu\rho}{}^{mn}D_\sigma\psi_{\tau i}\epsilon^{\mu\nu\rho\sigma\tau}+c\kappa^{-2/3}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd}e^e{}_\tau\epsilon^{\mu\nu\rho\sigma\tau}_{abcde}\ , \tag{2.7}$$

where $\epsilon^{\mu\nu\rho\sigma\tau}_{abcde}$ denotes the products of $\epsilon^{\mu\nu\rho\sigma\tau}$ and ϵ_{abcde}, and where $R_{\mu\nu ab}$ and D_σ all depend on the spin connection ω defined in (2.3). The peculiar powers of κ come about because in $d=5$ we have

$$[e]=0,\ \ [B]=\tfrac{3}{2},\ \ [\psi]=2,\ \ \text{and}\ [\kappa]=-\tfrac{3}{2}\ .$$

The complete variation of this ansatz reads

$$\begin{aligned}\kappa^{-1/3}\delta\mathscr{L}_{\mathrm{CS}}=&-\frac{i}{4}aR_{\mu\nu}{}^{ab}R_{\rho\sigma ab}(\bar{\epsilon}^i\psi_{\tau i})\epsilon^{\mu\nu\rho\sigma\tau}+2a(\delta\omega_\mu{}^{ab})R_{\rho\sigma ab}F_{\nu\tau}\epsilon^{\mu\nu\rho\sigma\tau}\\&-\frac{b}{8}(\bar{\epsilon}^i\{\gamma_{mn},\gamma_{ab}\}\psi_{\tau i})R_{\mu\sigma}{}^{ab}R_{\nu\rho}{}^{mn}\epsilon^{\mu\nu\rho\sigma\tau}+2ib\kappa(\bar{\epsilon}^i\gamma^a\gamma^{mn}\rho_{\sigma\tau i})R_{\nu\rho mn}\hat{F}_{a\mu}\epsilon^{\mu\nu\rho\sigma\tau}\\&+2b\kappa(\delta\omega_\nu{}^{mn})(\bar{\rho}_{\rho\mu}{}^i\gamma_{mn}\rho_{\sigma\tau i})\epsilon^{\mu\nu\rho\sigma\tau}+\frac{b}{4}\kappa(\delta\omega_\nu{}^{mn})(\bar{\psi}_\mu{}^i\{\gamma_{ab},\gamma_{mn}\}\psi_{\tau i})R_{\rho\sigma}{}^{ab}\epsilon^{\mu\nu\rho\sigma\tau}\\&+\frac{c}{2}(\bar{\epsilon}^i\gamma^e\psi_{\tau i})R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd}\epsilon^{\mu\nu\rho\sigma\tau}_{abcde}+4c(\delta\omega_\mu{}^{ab})R_{\rho\sigma}{}^{cd}\left[\tfrac{1}{4}\kappa(\bar{\psi}_\nu{}^i\gamma^e\psi_{\tau i})-\frac{i}{2}\epsilon_{\nu e\tau\alpha\beta}\hat{F}^{\alpha\beta}\right]\epsilon^{\mu\nu\rho\sigma\tau}_{abcde}\ .\end{aligned} \tag{2.8}$$

The expression in the large parentheses of the last term is due to the torsion of $\omega_{\mu mn}$:

$$2D_{[\mu}(\omega)e^m_{\nu]}=\frac{\kappa^2}{2}(\bar{\psi}_\mu{}^i\gamma^m\psi_{\nu i})-i\kappa\epsilon_{\mu m\nu ab}\hat{F}_{ab}\ . \tag{2.9}$$

The κ^0-order variations are

$$-\frac{i}{4}aR_{\mu\nu}{}^{ab}R_{\rho\sigma ab}(\bar{\epsilon}^i\psi_{\tau i})\epsilon^{\mu\nu\rho\sigma\tau}-\frac{b}{8}(\bar{\epsilon}^i\{\gamma_{mn},\gamma_{ab}\}\psi_{\tau i})R_{\mu\sigma}{}^{ab}R_{\nu\rho}{}^{mn}\epsilon^{\mu\nu\rho\sigma\tau}+\frac{c}{2}(\bar{\epsilon}^i\gamma^e\psi_{\tau i})R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd}\epsilon^{\mu\nu\rho\sigma\tau}_{abcde}\ . \tag{2.10}$$

It is easy to see that (2.10) vanishes if

$$b=\frac{i}{2}a=2c\ . \tag{2.11}$$

The sixth term in (2.8) is

$$\frac{b}{4}\kappa(\delta\omega_\nu{}^{mn})(\bar{\psi}_\mu{}^i\{\gamma_{ab},\gamma_{mn}\}\psi_{\tau i})R_{\rho\sigma}{}^{ab}\epsilon^{\mu\nu\rho\sigma\tau}=-\frac{b}{2}\kappa(\delta\omega_\mu{}^{ab})(\bar{\psi}_\nu{}^i\gamma_e\psi_{\tau i})R_{\rho\sigma cd}\epsilon^{\mu\nu\rho\sigma\tau}_{abcde}+b\kappa(\delta\omega_\mu{}^{ab})(\bar{\psi}_\nu{}^i\psi_{\tau i})R_{\rho\sigma ab}\epsilon^{\mu\nu\rho\sigma\tau}\ . \tag{2.12}$$

Using (2.11) the first part just supercovariantizes $F_{\nu\tau}$ in the second term of (2.8) and the second part just cancels the ψ torsion in the last term of (2.8). One is therefore left with

$$\begin{aligned}\kappa^{1/3}\delta\mathscr{L}_{\mathrm{CS}}=&\,2a(\delta\omega_\mu{}^{ab})R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon^{\mu\nu\rho\sigma\tau}-a\kappa(\bar{\epsilon}^i\gamma_{abc}\rho_{\sigma\tau i})R_{\nu\rho ab}\hat{F}_{c\mu}\epsilon^{\mu\nu\rho\sigma\tau}+2a\kappa(\bar{\epsilon}^i\gamma_a\rho_{\sigma\tau i})R_{\nu\rho ab}\hat{F}_{b\mu}\epsilon^{\mu\nu\rho\sigma\tau}\\&-2a(\delta\omega_\tau{}^{\mu\nu})R_{ab\rho\sigma}\hat{F}_{ab}\epsilon^{\mu\nu\rho\sigma\tau}+4a(\delta\omega_a{}^{\mu\nu})R_{\tau b\rho\sigma}\hat{F}_{ab}\epsilon^{\mu\nu\rho\sigma\tau}+ia\kappa(\delta\omega_\tau{}^{ab})(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab}\rho_{\rho\sigma i})\epsilon^{\mu\nu\rho\sigma\tau}\ .\end{aligned} \tag{2.13}$$

This is still the complete result for the variation of (2.7).

Up to order κ, we can neglect the torsion in $R_{\tau b\rho\sigma}$ in the fifth term of (2.13); therefore, this term vanishes by the cyclic identity. Moreover, the sixth term is omitted since it is of order κ^2. In the fourth term we replace $R_{ab\rho\sigma}$ by $R_{\rho\sigma ab}$. We will later show that the rest of the κ^1 variations in (2.13) cannot be canceled off shell. Let us first assume this and try to simplify (2.13) using the on-shell variation law of $\delta\omega_\mu{}^{ab}$. The gravitino field equation is

$$\gamma^{\mu\rho\sigma}\rho_{\rho\sigma i}\equiv R^\mu{}_i=0\ . \tag{2.14}$$

From this it easily follows that

$$\gamma_{mn}\rho_{mni} = \tfrac{1}{3}\gamma\cdot R_i = 0 \quad \text{and} \quad \gamma_b\rho_{abi} = -\tfrac{1}{2}R_{ai} + \tfrac{1}{6}\gamma_a(\gamma\cdot R_i) = 0 \; . \tag{2.15}$$

Multiplying (2.14) by $\gamma^{\nu\tau}$, one gets after a certain amount of γ-matrix algebra

$$\epsilon^{abcmn}\rho_{mni} + 6\gamma^{[a}\rho^{bc]}{}_i = 3\gamma^{[ab}R^{c]}{}_i - \tfrac{2}{3}\gamma^{abc}(\gamma\cdot R_i) = 0 \; . \tag{2.16}$$

Using this identity, the on-shell variation of $\omega_{\mu mn}$ becomes

$$\delta\omega_{\mu mn} = 2\kappa\bar{\epsilon}^i\gamma_{[m}\rho_{n]\mu i} \; . \tag{2.17}$$

The second term in (2.13) contains three γ matrices. We wish to reduce it to a one-γ term. We first prove the identity

$$(\bar{\epsilon}^i\gamma_{abc}\rho_{\underline{\sigma}\underline{\tau}i})R_{\nu\rho a\underline{b}}\hat{F}_{\underline{c}\underline{\mu}}\epsilon_{\mu\nu\rho\sigma\tau} = (\bar{\epsilon}^i\gamma_{abc}\rho_{c\tau i})R_{\nu\rho ab}\hat{F}_{\sigma\mu}\epsilon_{\mu\nu\rho\sigma\tau} \tag{2.18}$$

by applying the Schouten identity to the underlined indices. Then using (2.15) this further reduces to a one-γ term:

$$2(\bar{\epsilon}^i\gamma_a\rho_{b\tau i})R_{\nu\rho ab}\hat{F}_{\mu\sigma}\epsilon_{\mu\nu\rho\sigma\tau} \; . \tag{2.19}$$

The on-shell variations at order κ^1 can therefore be cast into the following form:

$$\kappa^{-1/3}\delta\mathscr{L}_{\rm CS} = 2a\kappa(\bar{\epsilon}^i\gamma_b\rho_{\mu ai})R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon_{\mu\nu\rho\sigma\tau} + 2a\kappa(\bar{\epsilon}^i\gamma_a\rho_{\sigma\tau i})R_{\nu\rho ab}\hat{F}_{b\mu}\epsilon_{\mu\nu\rho\sigma\tau} + 8a\kappa(\bar{\epsilon}^i\rho_{\mu\nu i})R_{\mu\nu ab}\hat{F}_{ab} \; . \tag{2.20}$$

It is remarkable, that the first two terms just have the same coefficients. Therefore one can apply the identity

$$(\bar{\epsilon}^i\gamma_b\rho_{\mu\nu i})R_{\rho ab\underline{a}}\hat{F}_{\sigma\tau}\epsilon_{\underline{\mu\nu\rho\sigma\tau}} = (\bar{\epsilon}^i\gamma_b\rho_{\mu ai})R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon_{\mu\nu\rho\sigma\tau} + (\bar{\epsilon}^i\gamma_a\rho_{\sigma\tau i})R_{\nu\rho ab}\hat{F}_{b\mu}\epsilon_{\mu\nu\rho\sigma\tau} \tag{2.21}$$

which follows from application of the Schouten identity to the underlined indices. This leads to

$$\kappa^{-1/3}\delta\mathscr{L}_{\rm CS} = -2a\kappa(\bar{\epsilon}^i\gamma_b\rho_{\mu\nu i})R_{\rho b}\hat{F}_{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau} + 8a\kappa(\bar{\epsilon}^i\rho_{\mu\nu i})R_{\mu\nu ab}\hat{F}_{ab} \; ,$$

where $R_{\mu aab} \equiv R_{\mu b}$.

Clearly the last term can be canceled by adding a term

$$-8ia\kappa^{4/3}(\det e)\hat{F}_{\mu\nu}R_{\mu\nu}{}^{mn}\hat{F}_{\rho\sigma}e^\rho{}_m e^\sigma{}_n \tag{2.22}$$

to the action. For the first one, we have two options. Since it is proportional to the Ricci tensor, which is part of the graviton field equation, one can in principle cancel it by modifying the vielbein transformation. But it can also be canceled by adding a term

$$8a\kappa^{7/3}(\bar{\rho}_{\mu\nu}{}^i\gamma^a\rho_{\lambda\rho i})\hat{F}_{\sigma\tau}e^\lambda{}_a\epsilon^{\mu\nu\rho\sigma\tau} \tag{2.23}$$

to the action. The variations of (2.22) and (2.23) contain of course also higher-order terms but these we will not consider. Therefore we conclude that by adding (2.22) and (2.23) to the original action $\mathscr{L}_{\rm CS}$ all variations of order κ^1 cancel on shell. In other words, on shell the CS term is supersymmetric to lowest and one-but-lowest order in κ.

III. OFF-SHELL VARIATIONS

In this section we will study the question whether the order-κ^1 off-shell variations of (2.13) can be canceled by only adding terms to the action. We therefore make a list of the most general terms one can add to the action which might cancel the order κ^1 off-shell variations of (2.13). The results are given in Table I. There are 11 terms with $\bar{\rho}\rho F$ and RFF structures, respectively. Let us define

$$\mathscr{L}' = 4\xi_1\mathscr{L}_1 + 4\xi_2\mathscr{L}_2 + 4\xi_3\mathscr{L}_3 + 8\xi_4\mathscr{L}_4 + 4\xi_5\mathscr{L}_5 + 8\xi_6\mathscr{L}_6 + 8\xi_7\mathscr{L}_7 + 4\xi_8\mathscr{L}_8 + \xi_9\mathscr{L}_9 + \xi_{10}\mathscr{L}_{10} + \xi_{11}\mathscr{L}_{11} \; . \tag{3.1}$$

Table II lists all the possible structures in the variations, and the coefficients of these terms in $\delta\mathscr{L}_{\rm CS}$ and $\delta\mathscr{L}'$, respectively. Requiring $\delta\mathscr{L}_{\rm CS}$ and $\delta\mathscr{L}'$ to cancel each other gives rise to an overdetermined system of 15 linear equations for 11 variables ξ_1–ξ_{11}. This system of equations has no solutions and consequently one cannot cancel the order κ^1 off-shell variations of $\delta\mathscr{L}_{\rm CS}$ by only adding terms to the action.

The on-shell invariance means that all order κ^1 terms in the variation are proportional to the gravitino field equation $R_{\mu i}$. Since the gravitino action is $-\frac{1}{2}\bar{\psi}^iR_i$, one can therefore always modify the gravitino transformation law such that $-\frac{1}{2}(\delta_{\rm extra}\bar{\psi}_\mu{}^i)R_{\mu i}$ cancel these terms. In this way one achieves off-shell invariance. The terms proportional to the gravitino field equation in the order κ' variations of $\mathscr{L}_{\rm CS}$ and (2.23) are given by

$$-a\kappa^{4/3}(\bar{\epsilon}^i\gamma_{b\mu}R_{ai})R_{\nu\rho ab}\hat{F}_{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau} - \frac{a}{2}\kappa^{4/3}(\bar{\epsilon}^i\gamma_{ab}R_{\mu i})R_{\nu\rho ab}\hat{F}_{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau} - \frac{a}{6}\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\nu\tau}R_{\mu i})R_{\rho\sigma ab}\hat{F}_{ab}\epsilon_{\mu\nu\rho\sigma\tau}$$
$$+\frac{4a}{3}\kappa^{4/3}(\bar{\epsilon}^i\gamma_\nu R_{\mu i})R_{\mu\nu ab}\hat{F}_{ab} + \frac{a}{3}\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu ab}\gamma\cdot R_i)R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon_{\mu\nu\rho\sigma\tau} \; . \tag{3.2}$$

In order to cancel these terms, the extra variation of the gravitino must be

$$\delta_{\text{extra}}\psi_\mu{}^i = -2a\kappa^{4/3}(\gamma_{ab}\epsilon^i)R_{\nu\rho b\mu}\hat{F}_{\sigma\tau}\epsilon_{a\nu\rho\sigma\tau} - a\kappa^{4/3}(\gamma_{ab}\epsilon^i)R_{\nu\rho ab}\hat{F}_{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau} - \frac{a}{3}\kappa^{4/3}(\gamma_{\nu\tau}\epsilon^i)R_{\rho\sigma ab}\hat{F}_{ab}\epsilon_{\mu\nu\rho\sigma\tau}$$
$$+\frac{8a}{3}\kappa^{4/3}(\gamma_\nu\epsilon^i)R_{\mu\nu ab}\hat{F}_{ab} - \frac{2a}{3}\kappa^{4/3}(\gamma_\mu\gamma_{\lambda ab}\epsilon^i)R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon_{\lambda\nu\rho\sigma\tau}\ . \tag{3.3}$$

Therefore we conclude that in order to cancel the order κ^1 variations it is not enough to only add terms to the action, but one is also forced to change the gravitino transformation law. Moreover, these corrections to the gravitino law are purely bosonic.

IV. THE $\bar{\rho}\rho\bar{\rho}\epsilon$ VARIATIONS

In the previous sections we have shown that the order-κ^1 variations in (2.13) can all be canceled on shell. In this section we will consider the last term

$$ia\kappa(\delta\omega_\tau{}^{ab})(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab}\rho_{\rho\sigma i})\epsilon^{\mu\nu\rho\sigma\tau}\ . \tag{4.1}$$

We will show that it vanishes on shell. (4.1) is an order-κ^2 term and it is easy to see that there is no term one can add to the action whose variation might cancel (4.1). The fact that (4.1) vanishes on shell gives hope that a supersymmetric extension might still exist at the complete κ^2 level, and it shows that there are also purely fermionic corrections to the $\delta\psi_\mu{}^i$ law in addition to the purely bosonic corrections given by (3.3). Using the on-shell variation of $\delta\omega_\tau{}^{ab}$ given by (2.17), up to a constant coefficient (4.1) becomes

$$(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab}\rho_{\rho\sigma i})(\bar{\epsilon}^j\gamma_a\rho_{b\tau j})\epsilon_{\mu\nu\rho\sigma\tau}\ . \tag{4.2}$$

We first apply (2.16) to $\rho_{\rho\sigma}\epsilon_{\mu\nu\rho\sigma\tau}$. This leads to

$$\begin{aligned}(4.2) &= -4(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab}\gamma_\mu\rho_{\nu\tau i})(\bar{\epsilon}^i\gamma_a\rho_{b\tau j}) - 2(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab}\gamma_\tau\rho_{\mu\nu i})(\bar{\epsilon}^j\gamma_a\rho_{b\tau j})\\ &= 8(\bar{\rho}_{\mu\nu}{}^i\gamma_b\rho_{\nu\tau i})(\bar{\epsilon}^j\gamma_\mu\rho_{b\tau j}) - 8(\bar{\rho}_{\mu\nu}{}^i\gamma_a\rho_{\nu\tau i})(\bar{\epsilon}^j\gamma_a\rho_{\mu\tau j}) - 2(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab\tau}\rho_{\mu\nu i})(\bar{\epsilon}^j\gamma_a\rho_{b\tau j})\\ &= -2(\bar{\rho}_{\mu\nu}{}^i\gamma_b\rho_{\nu\tau i})(\bar{\epsilon}^j\rho_{mnj})\epsilon_{\mu b\tau mn} - 4(\bar{\rho}_{\mu\nu}{}^i\gamma_b\rho_{\nu\tau i})(\bar{\epsilon}^j\gamma_b\rho_{\mu\tau j}) + (\bar{\rho}_{\mu\nu}{}^i\gamma_{mn}\rho_{\mu\nu i})(\bar{\epsilon}^j\gamma_a\rho_{b\tau j})\epsilon_{ab\tau mn}\ .\end{aligned} \tag{4.3}$$

Throughout the steps leading to (4.3) we have used the on-shell conditions (2.15) and (2.16) repeatedly. Using again (2.15) and (2.16), the first term in (4.3) becomes

$$\tfrac{1}{2}(\bar{\rho}_{\mu\nu}{}^i\rho_{\rho\sigma i})(\bar{\epsilon}^j\rho_{mnj})\epsilon_{\mu b\tau mn}^{b\nu\tau\rho\sigma} = -4(\bar{\rho}_{\mu\nu}{}^i\rho_{\nu\tau i})(\bar{\epsilon}^j\rho_{\mu\tau j}) \tag{4.4}$$

and for the last term in (4.3) we get

$$\begin{aligned}-2(\bar{\rho}_{\mu\nu}{}^i\gamma_m\gamma_n\rho_{\mu\nu i})(\bar{\epsilon}^j\rho_{mnj}) &= (\bar{\rho}_{\mu\nu}{}^i\gamma_m\rho_{ab i})(\bar{\epsilon}^j\rho_{mnj})\epsilon_{n\mu\nu ab} + 4(\bar{\rho}_{\mu\nu}{}^i\gamma_m\gamma_\mu\rho_{\nu ni})(\bar{\epsilon}^j\rho_{mnj})\\ &= 8(\bar{\rho}_{\mu\nu}{}^i\rho_{\nu\tau i})(\bar{\epsilon}^j\rho_{\mu\tau j})\end{aligned} \tag{4.5}$$

since $(\bar{\rho}_{\mu\nu}{}^i\gamma_m\rho_{abi})(\bar{\epsilon}^i\rho_{mnj})\epsilon_{n\mu\nu ab}=0$ by the Majorana property.

Finally we consider the middle term in (4.3). Let us define

$$\begin{aligned}S &\equiv (\bar{\rho}_{\mu\nu}{}^i\rho_{\nu\rho i})(\bar{\epsilon}^j\rho_{\mu\rho j})\ ,\\ V &\equiv (\bar{\rho}_{\mu\nu}{}^i\gamma_m\rho_{\nu\rho i})(\bar{\epsilon}^j\gamma_m\rho_{\mu\rho j})\ ,\\ T &\equiv (\bar{\rho}_{\mu\nu}{}^i\gamma_{mn}\rho_{\nu\rho}{}^j)(\bar{\epsilon}_j\gamma_{mn}\rho_{\mu\rho j})\ .\end{aligned} \tag{4.6}$$

They form a closed system using Fierz identities

$$\begin{aligned}8S &= S+V+T\ ,\\ 8V &= 5S-3V+T\ ,\\ T &= 5S+V\ .\end{aligned} \tag{4.7}$$

TABLE I. Terms whose variation might cancel the κ^1- order variations in (2.13).

$\mathscr{L}_1=\kappa^{7/3}(\bar{\rho}_{\mu a}{}^i\rho_{a\nu i})\hat{F}_{\mu\nu}$
$\mathscr{L}_2=\kappa^{7/3}(\bar{\rho}_{\mu a}{}^i\gamma_\nu\rho_{a\rho i})\hat{F}_{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau}$
$\mathscr{L}_3=\kappa^{7/3}(\bar{\rho}_{\mu\nu}{}^i\gamma_a\rho_{a\rho i})\hat{F}_{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau}$
$\mathscr{L}_4=\kappa^{7/3}(\bar{\rho}_{\mu\nu}{}^i\gamma_\rho\rho_{\sigma a i})\hat{F}_{a\tau}\epsilon_{\mu\nu\rho\sigma\tau}$
$\mathscr{L}_5=\kappa^{7/3}(\bar{\rho}_{\mu\nu}{}^i\gamma_{ab}\rho_{\mu\nu i})\hat{F}_{ab}$
$\mathscr{L}_6=\kappa^{7/3}(\bar{\rho}_{\mu a}{}^i\gamma_{\nu b}\rho_{\mu\nu i})\hat{F}_{ab}$
$\mathscr{L}_7=\kappa^{7/3}(\bar{\rho}_{ab}{}^i\gamma_{\mu\nu}\rho_{\mu\nu i})\hat{F}_{ab}$
$\mathscr{L}_8=\kappa^{7/3}(\bar{\rho}_{\mu a}{}^i\gamma_{\mu\nu}\rho_{b\nu i})\hat{F}_{ab}$
$\mathscr{L}_9=\kappa^{4/3}\hat{F}_{\mu\nu}R_{\mu\nu\rho\sigma}\hat{F}_{\rho\sigma}$
$\mathscr{L}_{10}=\kappa^{4/3}\hat{F}_{\mu\rho}R_{\mu\nu}\hat{F}_{\rho\nu}$
$\mathscr{L}_{11}=\kappa^{4/3}\hat{F}_{\mu\nu}R\hat{F}_{\mu\nu}$

TABLE II. Possible structures in the variations and the coefficients of $\delta\mathscr{L}_{CS}$ and $\delta\mathscr{L}'$.

Possible variations	Coefficients of $\delta\mathscr{L}_{CS}$	Coefficients of $\delta\mathscr{L}'$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu\nu}\rho_{\rho\sigma i})R_{\mu\nu\tau\rho}\hat{F}_{\sigma\tau}$	$4a$	$\xi_1+2\xi_3-2\xi_4+\xi_6-\xi_8$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu\nu}\rho_{\sigma\tau})R_{\nu\rho\sigma\tau}\hat{F}_{\rho\mu}$	$4a$	$-2\xi_2+2\xi_3+2\xi_4+4\xi_5+\xi_6$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\rho\tau}\rho_{\sigma\tau i})R_{\mu\nu\rho\sigma}\hat{F}_{\mu\nu}$	$4a$	$-2\xi_2+2\xi_3+2\xi_4-\xi_6-4\xi_7$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu\sigma}\rho_{\sigma\tau i})R_{\nu\tau}\hat{F}_{\mu\nu}$	$8a$	$-4\xi_2+4\xi_3+4\xi_4$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\rho\sigma}\rho_{\sigma\tau i})R_{\rho\mu}\hat{F}_{\mu\tau}$	$8a$	$4\xi_3-4\xi_4-2\xi_8$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu\nu}\rho_{\sigma\tau i})R_{\nu\sigma}\hat{F}_{\mu\tau}$	$8a$	$4\xi_3-4\xi_4+2\xi_6$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu\sigma}\rho_{\sigma\tau i})R\hat{F}_{\tau\mu}$	$4a$	$2\xi_3-2\xi_4$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{b}\rho_{\mu a i})R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon_{\mu\nu\rho\sigma\tau}$	$2a$	$-\xi_2-2\xi_3$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\mu}\rho_{abi})R_{\rho\sigma ab}\hat{F}_{\nu\tau}\epsilon_{\mu\nu\rho\sigma\tau}$	$-a$	$-\xi_5$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{a}\rho_{\sigma\tau i})R_{\nu\rho ab}\hat{F}_{b\mu}\epsilon_{\mu\nu\rho\sigma\tau}$	$2a$	$-\xi_3-\xi_4$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\tau}\rho_{\mu\nu i})R_{\rho\sigma ab}\hat{F}_{ab}\epsilon_{\mu\nu\rho\sigma\tau}$	$-a$	$-\xi_7$
$\kappa^{4/3}(\bar{\epsilon}^i\gamma_{\sigma}\rho_{a\tau i})R_{\nu\rho ab}\hat{F}_{b\mu}\epsilon_{\mu\nu\rho\sigma\tau}$	0	$-\xi_6$
$\kappa^{4/3}(\bar{\epsilon}^i\rho_{\mu\nu i})R_{\mu\nu ab}\hat{F}_{ab}$	$4a$	$2\xi_5-\xi_6+2\xi_7-i\xi_9$
$\kappa^{4/3}(\bar{\epsilon}^i\rho_{\mu\nu i})R_{\mu\rho}\hat{F}_{\nu\rho}$	$-8a$	$2\xi_6+2\xi_8-i\xi_{10}$
$\kappa^{4/3}(\bar{\epsilon}^i\rho_{\mu\nu i})R\hat{F}_{\mu\nu}$	$-2a$	$-2\xi_7-i\xi_{11}$

Solving these equations one gets $V=S$ and $T=6S$. Therefore the middle term in (4.3) is just

$$-4(\bar{\rho}_{\mu\nu}{}^{i}\rho_{\nu\tau i})(\bar{\epsilon}^{j}\rho_{\mu\tau j}) \ . \tag{4.8}$$

Adding (4.4), (4.5), and (4.8) together gives zero, and we conclude that also the $\bar{\rho}\rho\bar{\rho}\epsilon$ variation (4.1) vanishes on shell.

V. CONCLUSIONS

The action $\mathscr{L}_{CS}$ given by (2.7) can be made supersymmetric up to two orders in κ expansion provided (i) the coefficients a,b,c are fixed by (2.11), (ii) one adds terms (2.22) and (2.23) to the action $\mathscr{L}_{CS}$, and most importantly, (iii) one modifies the gravitino transformation law $\delta\psi_\mu{}^i$ to include the bosonic corrections given by (3.3). This strongly suggests that also in $d=10$ there are such extra terms in $\delta\psi_\mu$. They may still vanish in the Calabi-Yau background, but we intend to check this.

A further positive indication that a supersymmetric extension of the mixed Chern-Simons terms exists, is that the $\delta\mathscr{L}_{CS}\sim\bar{\rho}\rho\bar{\rho}\epsilon$ term at κ^2 order vanish on shell. It would be difficult to extend our results to higher orders in κ by the Noether method, but perhaps the geometrical methods discussed in the Introduction might help. It may be noted that these methods did give a very simple and general proof in $d=3$ dimensions that a supersymmetric extension of the gravitational Chern-Simons term did exist.[10,11]

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PART C

Introduction

The subject of quasi one–dimensional systems like polyacetylene and organic charge transfer salts has received a great deal of interest in recent years both because its experimental advances and its profound connection to the relativistic quantum field theories [27]. In a study of the spectrum for a one–dimensional spinless Fermi field coupled to a broken symmetry Bose field, Jackiw and Rebbi [28] first noted the occurence of a localized zero–energy solution ψ_0 to the Dirac equation when a topological soliton is present. The two degenerate states, corresponding to whether ψ_0 is occupied or not, carry charge $\pm 1/2$ measured in units of the charge possesed by elementary excitations of the unperturbed medium. Independently, Su, Schrieffer and Heeger [29] proposed a model (SSH model) for the quasi one– dimensional conductor polyacetylene $(CH)_x$ and noticed the formation of charge density wave due to the spontaneous symmetry breaking of the system, which leads to a two–fold degenerate vacuua and soliton formation. Just as in the Jackiw–Rebbi case, there is a localized solution in the presence of a soliton, with energy at the center of the gap. Thus, *if* one neglects the electron spin, the existence of the zero–energy state and fermion number $\pm 1/2$ are commen to both cases, leading to a fortunate convergence between condensed matter and reletivistic field theories.

However, electrons *do* posses spin degrees of freedom, therefore in this sense polyacetylene is not a experimental realization of the Jackiw–Rebbi soliton. In fact, topological excitations of the SSH model are neutral, spin 1/2 and charged spinless solitons respectively.

Rice and Mele [30] proposed a model for quasi one–dimensional organic charge transfer salts $(NMP)_x(Phen)_{1-x}TCNQ$ and $Q_n(TCNQ)_2$ which does seem to realize the Jackiw–Rebbi soliton. The model they studied is a quarter–filled Hubbard–Peierls model with infinite on–site electron–electron repulsion ($U = \infty$). In this case electrons can never hop across each other, the spin ordering is therefore a constant of motion. The Hilbert space is splitt into 2^M sectors corresponding to the different spin orderings (M being the number of the electrons) , and within a given sector, the electrons form a half–filled band of spinless fermions, realizing precisely the Jackiw–Rebbi model.

However, U is never infinite in realistic systems, and it is necessary to study the effect of perturbations of the order of $1/U$, since perturbations at this order break the 2^M fold degeneracy of the system. In paper **5**, the soliton excitations in both the small and large U limits are studied. It is found that as U varies from zero to infinity, the creation energy and the profile of the solitons change correspondingly, but the charge and the spin quantum numbers remain the same. Therefore, in contrast to the original expectation of Rice and Mele, the large U system resembles more the $U = 0$ system, but is qualitatively different from the infinite U system.

Having studied the strictly one–dimensional system at zero temperature, we proceed the discussion to the case where weak three–dimensional couplings are present. In this case, the ground state is two–fold degenerate, and naive couting arguments suggest that the solitons have charge $\pm 1/2$ and spin $\pm 1/4$! However, we observe that this only represent the average values, in fact, the spin of the solitons is not a sharp quantum observable due to the presence of long ranged spin wave fluctuations.

Charge versus Fermion Fractionalization in the Quarter-Filled Hubbard-Peierls Model

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We investigate the soliton excitations in the quarter-filled Hubbard-Peierls model in both the large- and small-U limits. For a strictly one-dimensional system at zero temperature, we find that the solitons in both limits are in one-to-one correspondence. In the presence of weak three-dimensional coupling, the large-U system differs qualitatively from the small-U system in that the spin associated with the solitons ceases to be a sharp quantum observable. We suggest a natural explanation of both the magnetic and the dielectric response measured in $(\mathrm{NPM})_x(\mathrm{Phen})_{1-x}\mathrm{TCNQ}$ [(N-methylphenazinium)$_x$(phenazine)$_{1-x}$(tetracyanoquinodimethanide)] and $\mathrm{Qn(TCNQ)_2}$ (quinolinium ditetracyanoquinodimethanide).

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It is now well established that a topological defect or kink may carry fractional or even irrational charge measured in units of the charge possessed by elementary excitations of the unperturbed medium[1-4]; moreover it has been demonstrated that this fractional charge is a sharp quantum observable.[5-7] The original work of Jackiw and Rebbi[1] suggested a further special possibility: If there exists a charge-conjugation symmetry in the presence of the soliton, then the kink should have two degenerate charge-conjugate states with fermion number $F=\pm\frac{1}{2}$. However, there is not yet an undisputed example of this situation in a realistic model. Indeed, it has been speculated that no real system could have such properties as long as charge-conjugation symmetry continues to hold in the presence of the kink.[8]

Relevant to this debate is the suggestion[9] that the elementary degrees of freedom of a quarter-filled Peierls-Hubbard system in the infinite-coupling limit ($U=\infty$) behave like spinless fermions of a half-filled band in the zero-coupling limit ($U=0$), and that therefore there are two charge-conjugate kink states with charge $Q=\pm\frac{1}{2}e$. Moreover, this large-U Peierls-Hubbard model is believed to be realized in certain charge-transfer salts[10] such as $(\mathrm{NMP})_x(\mathrm{Phen})_{1-x}\mathrm{TCNQ}$ [(N-methylphenazinium)$_x$(phenazine)$_{1-x}$tetracyanoquinodimethanide)] and $\mathrm{Qn(TCNQ)_2}$ (quinolinium ditetracyanoquinodimethanide).

At $U=\infty$, the ground state is 2^M-fold degenerate where M is the number of electrons, since different spin configurations all share the same energy. It is therefore necessary to study the limit as $U\to\infty$ in order to make contact with realistic systems. We have studied the system over the range from $U=0$ to $U\to\infty$. In this paper we summarize the results which will be reported in detail in a forthcoming publication.[11] In the $U\to\infty$ limit, the effective Hamiltonian which governs the spin excitations is that of a spin-$\frac{1}{2}$ antiferromagnetic spin chain. On the basis of this effective Hamiltonian, we argue that the ground state has a spontaneously broken translational symmetry which consists of a lattice dimerization driven by a half-filled band of spinless fermions, *and* a further much weaker dimerization of the dimers, driven by a spin-Peierls instability. The ground state thus has the same symmetry as for $U=0$. Moreover, we find that the solitons of this $U\to\infty$ double dimerized system have the same quantum numbers as those of the $U=0$ system though their profiles and relative creation energies are quite different. For all U we find three types of solitons: a spin-$\frac{1}{2}$, neutral, amplitude soliton $S_0^{1/2}$, a spinless phase soliton with $Q=\pm\frac{1}{2}e$, $S_{1/2}^0$, and a spin-$\frac{1}{2}$, $Q=\pm\frac{1}{2}e$, mixed phase and amplitude soliton, $S_{1/2}^{1/2}$. The neutral soliton is self-conjugate, but the degenerate charged-soliton doublets are not. For U large and $T\neq 0$, the spin-Peierls instability is very weak, and is likely to be suppressed in many experimentally relevant cases. Thus, we conclude this paper by analyzing the model in the absence of this distortion and its relation to experiments.

The Peierls-Hubbard model is defined by the Hamiltonian

$$H=-\sum_{n=1}^{N}\sum_{s=\pm\frac{1}{2}}[t_0-\alpha(u_n-u_{n+1})](c_{n,s}^\dagger c_{n+1,s}+\mathrm{H.c.})+\frac{K}{2}\sum_{n=1}^{N}(u_n-u_{n+1})^2+U\sum_{n=1}^{N}c_{n,\frac{1}{2}}^\dagger c_{n,\frac{1}{2}}c_{n,-\frac{1}{2}}^\dagger c_{n,-\frac{1}{2}}, \quad (1)$$

where $c_{n,s}^\dagger$ creates an electron of spin s on the latice site n, and u_n denotes the displacement of the nth lattice site. We treat u_n as a classical field. t_0, α, K, and U are coupling constants and N is the number of lattice sites.

At $U=0$, this model is identical to the Su-Schrieffer-Heeger (SSH) model,[4] except that the electron band is only quarter-filled, i.e., $k_\mathrm{F}=\pi/4a$, where a is the lattice constant. According to Peierls's theorem, the ground state is a charge-density-wave (CDW) state with period $2k_F$. In contrast to the half-filled SSH model, both the amplitude *and* the phase of the CDW condensate play a dynamical role in this system.[12] For instance, we can define the dimensionless order parameter by

$$\Delta_1(n)=(\sqrt{2}\alpha/t_0)u_n\cos(\pi n/2),\quad \Delta_2(n)=(\sqrt{2}\alpha/t_0)u_n\sin(\pi n/2),$$

or by

$$\Delta(n)e^{i\theta(n)}=\Delta_2(n)+i\Delta_1(n). \tag{2}$$

The various kinds of soliton excitations can be studied by consideration of the continuum limit of the $U=0$ discrete model (1):

$$\frac{H}{t_0}=\sum_s\int dx\,\psi_s^\dagger(x)[i\sigma_z\sqrt{2}a\,\partial_x+\Delta(x)\sigma_x\exp(i\sigma_z\theta(x))]\psi_s(x)+\int\frac{dx}{a}\left[\frac{\Delta^2(x)}{4\lambda}-A\Delta^4(x)\cos4\theta(x)\right], \tag{3}$$

where

$$\psi(x)=\begin{bmatrix}\psi_L(x)\\ \psi_R(x)\end{bmatrix}$$

represents the electronic states near k_F and $-k_F$, respectively, and $\lambda=\alpha^2/Kt_0$ is the dimensionless coupling constant. The last term arises from the *umklapp process* and is the source of the nontrivial phase dependence of the effective potential.[12,13] In the small-Δ limit, $A=(1/2\pi)\ln\Delta_0$.[13] For uniform order parameter, one finds $\Delta=\Delta_0=2\sqrt{2}Wa\exp(-\pi/2\sqrt{2}\lambda)$, where W is the momentum cutoff in the continuum model.

There are four degenerate ground states A,B,C, and D with $\Delta(x)=\Delta_0$ and $\theta(x)=0$, $\pi/2$, π, and $3\pi/2$, respectively. The various solitons are domain walls between degenerate phases, and are partially characterized by the change in the phase $\Delta\theta$ of the order parameter. For instance, the phase boundary between A and B phases is a $\Delta\theta=\pi/2$ soliton, while that between A and C phases is a $\Delta\theta=\pi$ soliton. We have studied the nature of these solitons both by numerically solving the discrete model to find the lattice configuration which minimizes the soliton creation energy and by approximate solution of the continuum model (see also Zhang, Kivelson, and Goldhaber[11] and Hubbard and Ohfuti and Ono[14]). The results can be summarized as follows. (1) There is one localized state associated with the $\pi/2$ soliton. If the state is unoccupied, the soliton has charge $Q=\frac{1}{2}|e|$ and spin 0, and is a pure phase soliton with $\Delta(x)\approx\Delta_0$ and $\theta(x)\approx\tan^{-1}\exp(x/l)$, where $l=a/\Delta_0^2(2A)^{1/2}$. Its creation energy is $E_s/t_0=(2A)^{1/2}\Delta_0^2+o(\Delta_0^3)$. If the state is singly occupied, the soliton has spin $-\frac{1}{2}$ and charge $Q=-\frac{1}{2}|e|$. It has mixed phase and amplitude character with

$$\Delta(x)\approx\Delta_0\tanh(x/\xi_0),\quad \theta(x)\approx\tan^{-1}\exp(x/l),$$

and creation energy $E_s/t_0=2\Delta_0/\pi+(2A)^{1/2}\Delta_0^2+o(\Delta_0^3)$, where $\xi_0=a/\Delta_0$ is the correlation length. If the state is doubly occupied, the soliton has creation energy of order Δ_0, and hence it is unstable with respect to formation of a topologically equivalent multiplet of three $\Delta\theta=-\pi/2$ pure phase antisolitons. (2) There is also one localized state associated with the $\Delta\theta=\pi$ soliton. The soliton is only stable if it is singly occupied, in which case it has $Q=0$ and spin $\frac{1}{2}$. This soliton is a pure amplitude soliton, and is precisely analogous to the neutral soliton in polyacetylene. An exact solution of the continuum model for this case gives $\Delta(x)=\Delta_0\tanh(x/\xi_0)$, $\theta(x)=0$ for its profile and $E_s/t_0=2\Delta_0/\pi$ for its creation energy.

Having identified the stable soliton excitations at $U=0$, we now proceed to study the limit $U\to\infty$ in (1). Rice and Mele[9] noticed that at $U=\infty$, the electrons can only singly occupy the sites, and two electrons cannot cross each other. It follows that the spin configuration is a constant of motion and the Hilbert space splits into 2^M disjoint subspaces, each with a definite spin configuration. Within each subspace, the electrons behave effectively like noninteracting fermions. In this case, the band of spinless fermions is half-filled, i.e., $k_F=\pi/2a$. The lattice will dimerize and open a gap in the electronic spectrum about the Fermi level. The ground state corresponds to a completely filled valence band, which can be represented equivalently as a state in which all the Wannier states $|R_n\rangle$ are occupied,[7] where $|R_n\rangle$ is exponentially localized about the center of the strong bond at position R_n. The ground state of a definite spin ordering can be represented by

$$|\Omega,\sigma_1,\ldots,\sigma_M\rangle=c\sum_{n_1<\cdots<n_M}F_{n_1\ldots n_M}^{R_1\ldots R_M}c_{n_1,\sigma_1}^\dagger\cdots c_{n_M,\sigma_M}^\dagger|0\rangle, \tag{4}$$

where $c_{n_j,\sigma_j}^\dagger$ creates the jth electron from the left at site n_j with spin σ_j. F is the determinant of the matrix W_{ij}, where $W_{ij}=\langle n_i|R_{m_j}\rangle$ is the Wannier function and c is a normalization constant.

At finite U, the effective Hamiltonian mixes the different spin configurations, since two electrons can occupy the same site as a virtual state. By straightforward degenerate perturbation theory we obtain the matrix elements of the effective Hamiltonian H_{eff} between the degenerate ground states (5). The resulting effective Hamiltonian can be cast into the form of a one-dimensional spin-$\frac{1}{2}$ Heisenberg chain,

$$H_{\text{eff}} = \sum_{i=1}^{M} J_i (\mathbf{S}_i \cdot \mathbf{S}_{i+1} - 1), \tag{5}$$

where

$$J_i = \frac{c^2}{U} \sum_{n_1 < \ldots < n_M} t_{n_i} t_{n_{i+1}} \delta_{1+n_i, n_{i+1}} F^{R_1 \ldots R_M}_{n_1 \ldots n_M} F^{R_1 \ldots R_i R_{i+1} \ldots R_M}_{n_1 \ldots n_i - 1 n_i \ldots n_M} + \frac{c^2}{U} \sum_{n_1 < \ldots < n_M} t^2_{n_i} \delta_{1+n_i, n_{i+1}} (F^{R_1 \ldots R_M}_{n_i \ldots n_M})^2. \tag{6}$$

In the case where the lattice is perfectly dimerized, J_i is actually independent of i. However, as we shall argue later, the lattice is not simply dimerized, but is doubly dimerized because of the spin-Peierls transition.[15]

It is a bit difficult to study this transition in general since J_i is rather complicated because the nth spin can only be loosely associated with the nth strong bond R_n. However, the model is quantitatively unchanged if we study it in the extremely dimerized limit where the nth spin *is* localized on the nth strong bond. Thus, we consider the limit in which the hopping matrix elements between the strong bonds t_s are much larger than that between the weak bonds t_w, so that we can treat t_w as a perturbation. To the zeroth order in t_w, electrons can only hop between the sites connected by the strong bonds. To second order in t_w, the ground-state spin degeneracy is removed by the hopping between the weak bonds. The resulting effective Hamiltonian is of the same form as Eq. (5), but with

$$J_i = \frac{2t_s^2}{U} t^2_{w,i} \left[\frac{1}{(U-\epsilon_+)(t_s+\epsilon_+)} + \frac{1}{(U-\epsilon_-)(t_s+\epsilon_-)} \right], \tag{7}$$

where $\epsilon_\pm = \frac{1}{2}[U \pm (U^2 + 16t_s^2)^{1/2}]$ and $t_{w,i}$ is the weak bond between the ith and the $(i+1)$th strong bond.

Because of the spin-Peierls transition, the weak bonds also dimerize in the ground state to form alternating weak and very weak bonds [Fig. 1(a)]. The various defects can be analyzed in the same way as before; in particular, we identify the three stable solitons as the $Q = \frac{1}{2}|e|$, $S=0$ soliton [Fig. 1(b)], the $Q = -\frac{1}{2}|e|$, $S = \frac{1}{2}$ soliton [Fig. 1(c)], and the $Q=0$, $S=\frac{1}{2}$ soliton [Fig. 1(d)].

Therefore, in the extremely dimerized limit, for any U the soliton quantum numbers are in one-to-one correspondence with those of the $U=0$ limit. There is no phase transition at $T=0$ and finite U. However, at large U, the spin-Peierls ordering becomes very weak. Even a small three-dimensional coupling is enough to destroy the spin-Peierls ordering. Moreover, since the characteristic energy of spin excitations is so small, even within a strictly one-dimensional model there is a large range where the temperature T is large compared to the creation energy of the neutral soliton, and hence the spin-density $4k_F$ ordering is completely destroyed, yet T is still small compared to any of the charged-soliton creation energies. Thus, it is interesting to consider the excitations of the large-U system in the absence of a $4k_F$ (double-dimerized) distortion. In this case the ground state is twofold degenerate, and there is only one type of soliton. Simple counting arguments suggest that this soliton has $Q = \pm\frac{1}{2}|e|$ and spin $\pm\frac{1}{4}$. Thus, one might conclude that there is actually half of an electron associated with the soliton!

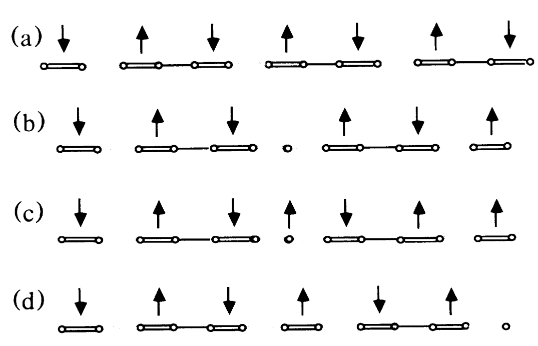

FIG. 1. The ground state and various soliton configurations in the extremely dimerized limit. Double lines, single lines, and broken lines represent strong bonds, weak bonds, and very weak bonds, respectively. Electrons localized on the strong bonds are represented by up and down arrows according to their spins.

This counting argument is correct as far as the expectation value of the spins is concerned. However, although the charge of the soliton is a sharp quantum observable,[5,6] the spin is not. To see this, we define the spin associated with the soliton as $\mathbf{S} = \sum_n f(n)\mathbf{s}(n)$ where $\mathbf{s}(n)$ is the spin density operator and $f(n)$ is a sampling function which is 1 over a region of size L about the soliton, and falls to zero beyond it. The mean square fluctuation of the spin can be computed easily from the spin-spin correlation function $G(n,m) = \langle \mathbf{s}(n) \cdot \mathbf{s}(m)\rangle - \langle \mathbf{s}(n)\rangle \cdot \langle \mathbf{s}(m)\rangle$ according to

$$\langle \Delta \mathbf{S}^2 \rangle = \frac{1}{2} \sum_{n,m} [f(n) - f(m)]^2 G(n,m). \tag{8}$$

In the presence of the soliton, $G(n,m) = G^0(n-m) + F(n,m)$, where G^0 is the correlation function of the perfect spin chain and $F(n,m) \sim G^0(n) G^0(m)$ is the correction due to the presence of the soliton. In the spin-Peierls state $G^0(n)$ is exponentially localized as a result of the gap in the spin-wave spectrum. Thus, for L sufficiently large, the fluctuations associated with the soliton spin are exponentially small. Without the spin-Peierls order, however, $G^0(n) \sim 1/|n|$, and hence the fluctuations are divergent. No particular spin can be associated with the soliton!

We note that in both $(\mathrm{NMP})_x(\mathrm{Phen})_{1-x}\mathrm{TCNQ}$ and $\mathrm{Qn(TCNQ)_2}$, the x-ray scattering data show the presence of a $4k_\mathrm{F}$ distortion, but no $2k_\mathrm{F}$ distortion.[10] We thus conclude that these materials are well described by a large-U quarter-filled Peierls-Hubbard model with the spin-Peierls distortion supressed. Experiments by Epstein *et al.*[10] on the magnetic susceptibility have been interpreted in terms of a weakly disordered Heisenberg spin chain, with a defect concentration proportional to the deviation in x from $x=0.5$ (the commensurate value). This has a natural interpretation in terms of a concentration of solitons proportional to $|x-0.5|$. This interpretation is lent further support by the fact that the x-ray scattering shows commensurate lock-in for a finite range of x about $x=0.5$. Experiments on the dielectric response and conductivity have been interpreted in terms of rather mobile, metallic, highly one-dimensional charged carriers in the presence of disorder.[15] The fact that the charge response of the system is characteristic of rather mobile electrons and the spin response is characteristic of an insulator is striking. It has a natural explanation in our model in terms of the almost complete decoupling between charge carriers (solitons) and the spin degree of freedom which occurs when the spin-Peierls transition is suppressed. We will discuss this point in greater detail in a future communication.

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